THE PRINCIPLES OF FINANCIAL AND STATISTICAL MATHEMATICS

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BY

MAXIMILIAN PHILIP, Sc.D., C.P.A.

PROFESSOR OF MATHEMATICS IN THE SCHOOL OF BUSINESS AND CIVIC ADMINISTRATION OF THE COLLEGE OF THE CITY OF NEW YORK

REVISED EDITION

NEW YORK
PRENTICE-HALL, INC.

FIRST EDITION

First printing	November, 1932
Second printing	January, 1934
Third printing	January, 1935
Fourth printing	February, 1936
Fifth printing	September, 1936
Sixth printing	June, 1937
Seventh printing	February, 1939
Eighth printing	August, 1940

REVISED EDITION

Ninth printing	September, 1941
Tenth printing	November, 1941
Eleventh printing	April, 1942
Twelfth printing	August, 1946
Thirteenth printing	January, 1947
Fourteenth printing	April, 1947

PRINTED IN THE UNITED STATES OF AMERICA

TO

MY SON

HERMAN

PREFACE

After using the former edition of *The Principles of Financial and Statistical Mathematics* as a text for about ten years, the author deemed it desirable to make drastic changes. Some topics have been simplified, others have been amplified, and their order has been changed. Some of the material in the former edition has been omitted, and considerable new material has been added. The general plan of having three distinct parts, each part suitable for the work of one semester, has, however, been retained. Throughout the book the emphasis is on logical reasoning rather than on the memorizing of formulas.

The book deals with problems that arise in two phases of business activity, investments and statistics. Investments, usually in bonds, mortgages, machinery, buildings, or life insurance, are made for long periods of time, and, because the time element is an important factor, compound interest is involved in all investment calculations. Statistics, systematized numerical facts gathered from past experience, may, when analyzed mathematically, be important guides in making intelligent plans for the future.

As presented in this book, the solutions of problems in investments and in statistics require a number of special symbols, but the mathematics involved is not difficult. The prerequisites for the study of these subjects consist of a knowledge of arithmetic, simple geometric relations, and elementary algebraic methods including logarithms. It is essential, however, that the student be able to recognize and to use these elementary mathematical relations with facility in order to appreciate the beauty and the simplicity of the techniques.

In order to avoid the necessity for reference to other texts, the required elementary basic mathematics is included. The entire subject matter in the book is accordingly presented in three parts:

I. Basic Mathematics;
II. Financial Mathematics;
III. Statistical Methods.

Part I begins with arithmetic methods of solving business problems, special attention being given to simple interest, which is not so simple as the name might imply. Elementary algebraic methods are presented, including the solution of simultaneous linear equations and of quadratic equations, and the use of logarithms. The simple geometric relations discussed are necessary for a proper understanding of graphic representation and of linear interpolation.

A student who enters college with the usual requirements in mathematics should find no difficulty in reviewing the contents of Part I without help. A student whose mathematical equipment is inadequate should study Part I as regular class work for one semester.

Part II begins with the meaning and use of the four compound interest and annuity symbols s_i^n, v_i^n, $s_{\overline{n}|i}$, and $a_{\overline{n}|i}$. In the application of these four symbols two fundamental principles are emphasized.

(1) The Principle of Equivalent Obligations is used repeatedly in setting up the equation necessary for the solution of an investment problem. The calculation of life insurance premiums and reserves is an application of this principle.

(2) The Principle of Equivalent Interest Rates obviates the use of discouraging complex formulas usually found in texts on investment calculations.

In addition to these two principles, a special device is employed. This device consists of constructing a hypothetical obligation whose value is known, in order to find the unknown value of a given obligation. By this device a seemingly difficult problem such as the valuation of serial bonds is reduced to a very simple calculation.

Additional mathematical topics included in Part II are methods of abbreviated multiplication and division, the binomial theorem, and the method of finite differences. The last two topics are presented briefly but in sufficient detail to enable the student to make calculations that are more accurate than are possible by the use of linear interpolation or logarithms.

A student who has mastered the contents of Part II should find no difficulty in solving almost any problem in investments with which he may be confronted.

Part III is an *introduction* to the mathematics of statistics and does not pretend to be exhaustive. A thorough study of statistics requires a sound knowledge of the Differential and Integral Calculus. The author hopes that this introduction will be instrumental in creating a desire for such further study.

Part III begins with a discussion of probability. In many cases, probabilities are indicated by the terms of the expansion of the binomial $(p + q)^n$, and, since these terms may be represented graphically by points, probability curves follow naturally. A derivation of the equation of the normal probability curve, $y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$, by elementary mathematical methods is also given.

A study of the different forms into which the equation of a curve may be transformed by the use of different scales serves to familiarize the student with a wide variety of curves useful in the study of statistics. Finally, the con-

verse problem of finding an approximate equation of a curve that will best fit a given table of corresponding values is solved by the method of least squares. The normal equations necessary for a least square solution are obtained with the aid of the quadratic trinomial $At^2 + Bt + C$. Special emphasis is given to the reasons for using the method of least squares and to the tests for the goodness of fit.

Parts I, II, and III are about equal in length, and each part contains sufficient material for the work of one college semester at three hours a week. The student who is properly prepared should find no difficulty in completing the book in one college year.

The extensive and well-known Glover's "Tables of Applied Mathematics," published by Mr. George Wahr, form the basis for most of the tables accompanying this text. Permission to use these tables is gratefully acknowledged. There are 17 tables, some of which are set forth on one page and some on two pages that face each other. The tables are bound separately in a small pamphlet, thus enabling the student to use them more conveniently than would be possible if the table and text were bound together.

The author acknowledges with grateful appreciation the many helpful aids and suggestions he received during the preparation of this revision from the members of the Department of Mathematics at the School of Business and Civic Administration of The City College who have been using the former edition as a text. Special thanks are due to Russell D. Loucks, Robert K. Stranathan, Charles C. Grove, Edwin A. Hill, and Austin J. Bonis for their keen and constructive comments.

MAXIMILIAN PHILIP

The City College, New York

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PART I

BASIC MATHEMATICS

CHAPTER I

ARITHMETIC CALCULATIONS

1. Indicated operations. Tedious arithmetic calculations may often be avoided by first indicating the operations that are to be performed, because some operations nullify or cancel other operations. The signs for the ordinary operations of arithmetic are: + (plus) for addition, − (minus) for subtraction, × (times) for multiplication, ÷ (divided by) for division.

7 + 8 means that 8 is to be added to 7. The numbers 7 and 8 are called *addends* and the result, 15, is called the *sum*. The plus sign is sometimes omitted. Thus $6\frac{2}{3}$ means $6 + \frac{2}{3}$; 5.7 means 5 + .7; 387 means 300 + 80 + 7.

19 − 2 means that 2, the *subtrahend*, is to be subtracted from 19, the *minuend*. The result, 17, is the *difference*.

6 × 3 means that 6 and 3, called *factors*, are to be multiplied. The result, 18, is the *product*. When several numbers are to be multiplied, dots replace the times signs, provided no ambiguity can arise by mistaking a dot for a decimal point. Thus 8·7·5 means 8×7×5. Even the dot may be omitted if the factors are enclosed within parentheses. Thus (63) (47) means 63 × 47.

There is a special form for writing the product of two or more equal factors.

5 × 5 is written 5^2, read "5 squared," or "the square of 5";
7 × 7 × 7 is written 7^3, read "7 cubed," or "the cube of 7";
9 × 9 × 9 × 9 is written 9^4, read "9 fourth," or "the fourth power of 9."

The small figures, 2 in 5^2, 3 in 7^3, 4 in 9^4, are called *exponents*.

28 ÷ 7 means that 28, the *dividend*, is to be divided by 7, the *divisor*. The result, 4, is the *quotient*. The quotient is

also called the *ratio* of 28 to 7 and may be indicated by writing $\frac{28}{7}$. Whole numbers are called *integers*, and indicated divisions, such as $\frac{28}{7}$ or $\frac{5}{8}$, are called *fractions*. In the fraction form the dividend is called the *numerator*, and the divisor is called the *denominator*.

The sign of equality, = (equals), indicates that two number combinations have the same value.

$$8 + 97 - 97 = 8;\ 13 \times 37 \div 37 = 13;\ 2 \times 5^3 = 2 \times 125 = 250.$$

2. Sequence of indicated operations. In the sequence

$$19 + 3 \times 2 - 18 \div 3 + 15 \div 5 \times 4,$$

it is to be understood that multiplications and divisions must be performed in the order in which they appear, reading from left to right, *before* additions and subtractions are performed. With this understanding we obtain, in this example,

$$19 + 6 - 6 + 12,$$

and finally 31. If the operations are to be performed in a different order, parentheses are used. In that case the combinations in the parentheses must first be reduced to simple numbers in accordance with the rule stated.

Thus

$$\begin{aligned} &2 \times (19 + 3) - 18 \div (3 + 15 \div 5) \times 5 \\ &= 2 \times 22 - 18 \div 6 \times 5 \\ &= 44 - 15 = 29. \end{aligned}$$

Exercise 1

Verify each of the following:

1. $27 \div 9 + 3 \times 8 - 6 \div 2 = 24.$
2. $(27 \div 9) + (3 \times 8) - (6 \div 2) = 24.$
3. $27 \div 9 + 3 \times (8 - 6 \div 2) = 18.$
4. $27 \div (9 + 3) \times (8 - 6) \div 2 = 2\frac{1}{4}.$
5. $27 \div (9 + 3 \times 8 - 6) \div 2 = \frac{1}{2}.$

6. $16 - (2^3 + 7) \div 5 + 3 \times (5^2 - 9) = 61.$

7. $(16 - 2^3 + 7) \div 5 + (3 \times 5^2 - 9) = 69.$

8. $(16 - 2^3) \times (5^3 - 20 \times 6) - 2 \times (5 + 3 \times 2 - 7) = 32.$

9. $5^2 \times 3^2 - (3 \times 4 - 1)^3 \div 11 = 104.$

10. $(16 - 2 \times 7)^3 \div 2^3 + (2 \times 3^2 - 2^4)^3 = 9.$

3. Letters. If the letters a, b, c, d, and so forth represent numbers, the signs that indicate how they are to be combined are the same as in arithmetic except that the sign for multiplication is usually omitted. Thus $5ab = 5 \times a \times b = 5 \cdot a \cdot b$.

If we wish to indicate that a number consists of the three figures or *digits* a, b, and c, we may not write $a\ b\ c$, for this would mean $a \times b \times c$. The number is indicated by writing $100a + 10b + c$.

$$5ab + 7a^2 - \frac{3a}{b} + (a + b)(c - d)$$

means

$$5 \times a \times b + 7 \times a \times a - 3 \times a \div b + (a + b) \times (c - d).$$

The first form shows more clearly that multiplications and divisions should be performed before additions and subtractions unless parentheses appear.

A general statement, one that is true for all numbers, can be made by using words or by using letters to represent numbers. The statement, "The number of units in the area of a rectangular figure is found by multiplying the number of units in the length by the number of units in the width," is a *formula* which is written more concisely as $A = lw$.

Exercise 2

1. Find the value of $5ab + 7a^2 - \dfrac{3a}{b} + (a + b)(c - d)$:

(a) if $a = 10, b = 2, c = 8, d = 6$.

(b) if $a = 6, b = 3, c = 5, d = 2$.
(c) if $a = 2, b = 3, c = 9, d = 7$.

2. A number consists of two figures or digits, t and u.
(a) Indicate the number properly.
(b) Indicate the sum of the digits.
(c) Indicate the product of the digits.

3. Use any convenient letters and write formulas for the following:

(a) The profit is found by subtracting the cost from the selling price.

(b) John is 12 years older than Mary.

(c) The time in San Francisco is 4 hours earlier than the time in New York.

(d) The circumference of a circle is π (pi) times its diameter, where π is approximately $3\frac{1}{7}$.

(e) In any subtraction, the difference added to the subtrahend gives the minuend.

(f) When a division is performed, the dividend is always obtainable by multiplying the quotient by the divisor and adding the remainder.

(g) The result of multiplying the sum of two numbers by their difference is the same as the difference between the squares of the numbers. (Verify this statement.)

(h) The square of the sum of two numbers exceeds the sum of the squares of the two numbers by twice the product of the two numbers. (Verify this statement.)

(i) The square of the difference of two numbers is less than the sum of the squares of the two numbers by twice the product of the two numbers. (Verify this statement.)

(j) The square of the sum of two numbers exceeds the square of the difference between the same two numbers by four times the product of the two numbers. (Verify this statement.)

4. Indicated operations simplified. In a succession of additions and subtractions, we start with the number 0, to which additions are made. The numbers may be arranged in any order provided the + or the − sign before each number is carried along with the number.

$$8 + 5 - 7 - 2 = 0 + 8 + 5 - 7 - 2 = 0 + 5 + 8 - 2 - 7$$
$$= 0 + 8 - 7 + 5 - 2 = 0 + 5 - 2 + 8 - 7.$$

Two successive additions may be replaced by the addition of a single number which is the sum of the two. Two successive subtractions may be replaced by the subtraction of a single number which is the sum of the two.

$$18 - 7 + 9 - 12 = 0 + (18 + 9) - (7 + 12) = 0 + 27 - 19 = 8.$$

In a succession of multiplications and divisions, we start with the number 1, which is to be multiplied. The numbers may be arranged in any order provided the $\times$ or the $\div$ sign before each number is carried along with the number.

$$8 \times 15 \div 6 \div 5 = 1 \times 8 \times 15 \div 6 \div 5 = 1 \times 15 \div 5 \times 8 \div 6.$$

Two successive multiplications may be replaced by a single multiplication, the product of the two multipliers. Two successive divisions may be replaced by a single division, the product of the two divisors.

$$1 \times 8 \times 15 \div 6 \div 5 = 1 \times (8 \times 15) \div (6 \times 5) = 1 \times 120 \div 30 = 4.$$

5. Common fractions. The common fraction $\frac{3}{7}$ indicates that 3, the numerator, is to be divided by 7, the denominator.

(a) $$\frac{3}{7} = 3 \div 7 = 3 \div 7 \times 5 \div 5 = (3 \times 5) \div (7 \times 5) = \frac{3 \times 5}{7 \times 5}.$$

$$\frac{3 \times 4}{7 \times 4} = 3 \times 4 \div 7 \div 4 = 3 \div 7 \times 4 \div 4 = 3 \div 7 = \frac{3}{7}.$$

These two examples illustrate the principle: The value of a fraction is not changed if both the numerator and the denominator are multiplied or divided by the same number.

Thus

$$\frac{3}{5} = \frac{3 \times 2}{5 \times 2} = \frac{6}{10}, \quad \frac{16}{24} = \frac{16 \div 8}{24 \div 8} = \frac{2}{3}.$$

In general,

$$\frac{a}{b} = \frac{na}{nb} \quad \text{and} \quad \frac{ka}{kb} = \frac{a}{b}.$$

(b) $\frac{3}{7} \times \frac{5}{8} = 3 \div 7 \times 5 \div 8 = (3 \times 5) \div (7 \times 8) = \frac{3 \times 5}{7 \times 8}.$

The product of two fractions is a fraction whose numerator is the product of the two numerators and whose denominator is the product of the two denominators.

In general,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Two fractions such as $\frac{3}{7}$ and $\frac{7}{3}$ are called *reciprocals.* The product of the reciprocals $\frac{a}{b}$ and $\frac{b}{a}$ is 1, since $a \div b \times b \div a = a \div a \times b \div b = 1.$

(c) $\frac{3}{7} \div \frac{5}{8} = \frac{\frac{3}{7}}{\frac{5}{8}} = \frac{\frac{3}{7} \times 7 \times 8}{\frac{5}{8} \times 7 \times 8} = \frac{3 \times 8}{5 \times 7} = \frac{3}{5} \times \frac{8}{7}.$

To divide by a fraction, multiply by its reciprocal.

In general,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

(d) $\frac{3}{8} + \frac{2}{8} - \frac{1}{8} = \frac{3+2-1}{8} = \frac{4}{8} = \frac{1}{2}.$

$$\frac{6}{7} + \frac{5}{8} = \frac{6 \times 8}{7 \times 8} + \frac{5 \times 7}{8 \times 7} = \frac{48}{56} + \frac{35}{56} = \frac{83}{56}.$$

$$\frac{6}{7} - \frac{5}{8} = \frac{6 \times 8}{7 \times 8} - \frac{5 \times 7}{8 \times 7} = \frac{48}{56} - \frac{35}{56} = \frac{13}{56}.$$

A fraction can be added to or subtracted from another fraction by changing either or both fractions so that they have the same denominator.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}.$$

(e) The rule stated in Art. 2, page 4, applies to fractions as well as to whole numbers:

$$\frac{2}{3} - \frac{1}{2} \times \frac{1}{3} + \frac{1}{4} \div \frac{3}{5} = \frac{2}{3} - \frac{1}{6} + \frac{5}{12} = \frac{8 - 2 + 5}{12} = \frac{11}{12}.$$

Exercise 3

Simplify the following combinations:

1. $5\frac{1}{2} - 3\frac{1}{3} \times 6 \div 15$
2. $5\frac{5}{7} \div 3\frac{1}{3} + 1\frac{1}{5}$
3. $\frac{5}{9} \times \frac{6}{35} \div \frac{3}{14}$
4. $\frac{5}{9} + \frac{3}{5} - \frac{1}{2} - \frac{1}{3}$
5. $7\frac{1}{2} \div 1\frac{2}{3} \div 3 \times 2$
6. $(5\frac{1}{2} - 3\frac{2}{3}) \times 6 \div 15$
7. $5\frac{5}{7} \div (3\frac{1}{3} + 1\frac{1}{5})$
8. $3\frac{3}{4} \div (\frac{1}{2} + \frac{1}{3}) + \frac{1}{6}$
9. $3\frac{3}{4} \div \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$
10. $3\frac{3}{4} \div (\frac{1}{2} + \frac{1}{3} + \frac{1}{6})$
11. $1\frac{2}{3} + 2\frac{1}{6} - 1\frac{1}{2}$
12. $1\frac{2}{3} - 2\frac{1}{6} + 1\frac{1}{2}$
13. $\frac{3}{4} \times \frac{7}{12} \div 1\frac{3}{4} + 1\frac{1}{2}$
14. $\frac{3}{4} \times (\frac{7}{12} \div 1\frac{3}{4} + 1\frac{1}{2})$
15. $\frac{3}{4} \times \frac{7}{12} \div (1\frac{3}{4} + 1\frac{1}{2})$
16. $(1\frac{1}{2} + \frac{3}{4}) - (\frac{7}{12} \div 1\frac{3}{4})$

6. Decimal fractions. A common fraction may be changed into a decimal form as follows:

$$\frac{6}{7} = \frac{6.0}{7} = 0.8\frac{4}{7};\ \frac{6.00}{7} = 0.85\frac{5}{7};\ \frac{6.000}{7} = 0.857\frac{1}{7}.$$

The decimal .01 is also written 1%, read "1 per cent." Thus $\frac{6}{7} = 85\frac{5}{7}\% = 85.7\frac{1}{7}\%$; $.00\frac{2}{3} = \frac{2}{3}$ of 1%.

A decimal fraction or a number written as a percentage may be changed into a common fraction.

$$0.8\frac{4}{7} = \frac{8\frac{4}{7}}{10} = \frac{8\frac{4}{7} \times 7}{10 \times 7} = \frac{60}{70} = \frac{6}{7}.$$

$$16\tfrac{2}{3}\% = \frac{16\frac{2}{3}}{100} = \frac{16\frac{2}{3} \times 3}{100 \times 3} = \frac{50}{300} = \frac{1}{6}.$$

Sometimes a computation is simpler if a common fraction is changed into a decimal, and sometimes if a decimal is changed into a common fraction.

$$783 \times 375 = 783 \times \tfrac{3}{8} \times 1000 = \frac{2{,}349{,}000}{8} = 293{,}625.$$

$$783 \div 375 = \frac{783}{\frac{3}{8} \times 1000} = .783 \times \tfrac{8}{3} = \frac{6.264}{3} = 2.088.$$

$$376.58 \times \tfrac{3}{5} = 376.58 \times .6 = 225.948.$$

$$438.4 \times .125 = 438.4 \times \tfrac{1}{8} = 54.8.$$

Exercise 4

1. Change to 4 place decimals: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{7}{8}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{5}{9}$, $\frac{7}{16}$, $\frac{1}{72}$, $\frac{1}{73}$.

2. Change to common fractions in their simplest form: 0.25, 0.125, 0.375, $1\frac{1}{2}\%$, $6\frac{2}{3}\%$, $36\frac{2}{3}\%$, 0.485, 0.5%, $.00\frac{1}{3}\%$.

3. Perform the indicated operations using simple methods wherever possible:

573×625; $537 \div 625$; 5600×1250; $4800 \div 1250$; $7236 \times 16\frac{2}{3}\%$; $658 \div 14\frac{2}{7}\%$.

4. Find what per cent 6 is of 24; of 25; of 27; of 30; of 36; of 40; of 42; of 45; of 48; of 50.

5. Find the number if 60 is: 2% of the number; $\frac{3}{4}$ of the number; $3\frac{1}{3}\%$ of the number; $\frac{5}{7}$ of the number; $\frac{1}{2}$ of the number.

6. An article that cost $48 is sold for $54. Find what per cent the profit is: (a) of the cost; (b) of the selling price.

7. An article was bought for $100 and sold so that the profit was 10% of the selling price. Find the profit.

8. An article cost $50. It was marked for sale at $60. What per cent of the cost was the markup?

9. An article was marked for sale at $75 and was then marked down 20% of the first marked price. How much was the markdown?

10. An article cost $80. It was marked up 40% and was then marked down 25% of the first marked price and sold. Find what per cent the profit or loss was: (a) of the cost; (b) of the selling price.

11. An article was marked up $33\frac{1}{3}\%$ of cost. The price was then marked down $12\frac{1}{2}\%$, and finally the price was marked down 10%. If the article were sold at the last marked price, find what per cent the profit or loss was: (a) of the cost; (b) of the selling price.

12. An article was marked up 50% of cost. The price was then marked down 25%. A subsequent markdown resulted in a price which was the same as the cost. Find what per cent the last markdown was.

13. Fifty articles were bought at a cost of \$25 each. The price was marked up 40% and 10 were sold. The price was then marked down 20% from the previous price and 20 were sold. The last 20 were sold at 10% above cost. Find the total profit.

14. Anthony's salary for 1940 was 10% above his 1939 salary, which was 10% below his 1938 salary. Find the per cent increase or decrease from the 1938 to the 1940 salary.

15. 85% of a number is less than 92% of the number by 35. Find the number.

7. Trade discount. The term *discount* means a reduction from a stated price. A wholesale dealer often quotes a price to the retailer such as \$32 per dozen, less 25%, $16\frac{2}{3}$%, and 10%, meaning that from the quoted price of \$32 per dozen, the retailer may deduct 25%, leaving \$24; he may then deduct $16\frac{2}{3}$% or $\frac{1}{6}$ from \$24, leaving \$20; and finally he may deduct 10% from \$20, leaving a net price of \$18 per dozen.

A simpler procedure is to indicate the calculations by noting that, when 25% or $\frac{1}{4}$ is deducted, the remainder is $\$32 \times \frac{3}{4}$; and when $16\frac{2}{3}$% or $\frac{1}{6}$ is then deducted, the remainder is $\$32 \times \frac{3}{4} \times \frac{5}{6}$; and when the final 10% or $\frac{1}{10}$ is deducted, the remainder is

$$\$32 \times \frac{3}{4} \times \frac{5}{6} \times \frac{9}{10} = \$32 \times \frac{9}{16} = \$32 \times 56\tfrac{1}{4}\% = \$18.$$

From this indicated method of calculating the net price it follows that: (1) the order of discounts may be changed without causing any change in the net price; (2) the single discount equivalent to the successive discounts is easily found.

Thus if the discounts quoted had been $16\frac{2}{3}$%, 10%, and 25%, the net price would have been $\$32 \times \frac{5}{6} \times \frac{9}{10} \times \frac{3}{4}$ exactly as before. Since the net price is $\$32 \times 56\frac{1}{4}\%$, the single discount equivalent to the successive discounts is $100\% - 56\frac{1}{4}\%$, or $43\frac{3}{4}\%$.

Exercise 5

A quoted price of $500 is subject to discounts of 50%, 30%, and 10%.

1. What is the net price?

2. What single discount may replace the three discounts?

3. Would the net price be more or less if the discounts of 50% and 30% were replaced by a single discount of 80%?

4. Would the net price be more or less if the discount of 50% were replaced by the two discounts of 40% and 10%?

5. Would the net price be more or less if the discounts were replaced by discounts of 30%, 10%, and 50%?

6. If the net price is to remain unchanged but only two discounts are to be quoted, find the second discount if the first is: (a) 50%; (b) 30%; (c) 10%.

7. If the net price is to be $125 and two discounts are 50% and 30%, find the third discount.

8. If the net price is to be $250 and two discounts are 30% and 10%, find the third discount.

8. Time. A year, 365 days, is divided into 12 months, of which April, June, September, and November have 30 days each, February has 28 days, and the remaining 7 months have 31 days each. An additional day is added to February in leap years. A leap year occurs when the number of the year is divisible by 4, except in the case of century years, when the number of the year must be divisible by 400. Thus 1896 and 1600 were leap years, but 1930 and 1900 were not.

If a loan is made on March 5, for 3 months, it is due on June 5, but, if the loan is for 90 days, it is due exactly 90 days after March 5, or on June 3. If a loan made on October 18, 1939, is due March 10, 1940, the exact number of days for which the loan was made is found by counting the days that have elapsed, excluding the first date mentioned. The count gives 144 days. The ordinary elapsed time is

determined by writing March 10, 1940, as 1940 − 3 − 10 and October 18, 1939, as 1939 − 10 − 18. We now subtract, using 30 days to the month and 12 months to the year. The difference is 4 months and 22 days, or 142 days.

Exercise 6

1. A loan was made July 20. Find the date it is due if the loan was made for: (a) 1 month; (b) 2 months; (c) 3 months; (d) 30 days; (e) 60 days; (f) 90 days.

2. A loan was made November 17, 1939. Find the ordinary and the exact number of days if the loan was due: (a) Jan. 10, 1940; (b) March 5, 1940; (c) Feb. 17, 1940.

3. Which of the following were not leap years: 1800; 1812; 1888; 1898; 1900; 1914; 1916; 1932; 1940?

9. Cash discount. Jonah Kurb buys merchandise amounting to $500, the terms being indicated by 2/10, n/30. He understands this notation to mean that if he pays by the end of 10 days he may deduct a cash discount of 2% and settle his bill by a payment of $490. If he does not take advantage of the discount, the bill must be settled at the end of 30 days by a payment of $500. At the end of 10 days Kurb pays $300 for some of the items of the bill. How much should he pay at the end of 30 days?

Since for each dollar of the bill he need pay only 98 cents at the end of 10 days, a payment of $300 cancels $\frac{\$300}{.98}$ or $306.12 of the original bill, and Kurb still owes $193.88.

Exercise 7

1. On March 1, Albert Harper bought merchandise amounting to $1200, the terms being 2/10, n/30.

(a) How much should he pay to settle his bill on March 11?

(b) How much should he pay to settle his bill on March 20?

(c) If he pays $800 on or before March 11, how much of the original bill did he pay? How much is still due? On what date is the balance due?

2. Robert May bought merchandise Jan. 15, 1939, amounting to \$6800, the terms being 3/30, n/90. May took advantage of the terms offered to the extent of paying \$2800 in time to be allowed the 3% discount. How much should he pay at the end of 90 days? On what date is the balance due?

3. Henry Parr bought merchandise amounting to \$5000, the terms being 3/10, 2/30, n/90. He paid \$2000 at the end of 10 days, and \$2500 at the end of 30 days. How much should he pay at the end of 90 days?

10. Interest. Drew borrowed \$5000 from Carr and at the end of 6 months gave Carr \$5150 in settlement of the debt. The extra \$150, called *interest*, is money paid for the use of another person's money and depends upon: (1) the amount of the loan or the *principal*; (2) the *rate* of interest, stated as hundredths or per cent of the principal, for its use for one year; (3) the length of time that elapsed from the date when the loan was made to the date that it was repaid.

11. Simple interest. Simple interest is interest paid only on the original principal. It accumulates as a part of the debt but does not itself bear interest and is payable when the debt is paid.

The conditions of the loan may be that interest shall be paid at regular intervals of time. In that case, unpaid interest when due is added to the debt, and interest is calculated on the increased debt. The interest thus calculated, consisting partly of interest on interest, is called *compound interest* and will be discussed in Chapter VI. For the present, we shall confine our discussion to simple interest.

The simple interest, I, at rate r per year, on a principal p for n years, whether n is an integer or a fraction, is

$$I = prn.$$

Thus the simple interest on \$5000 for 6 months at 6% per year is

$$I = \$5000 \times .06 \times \tfrac{6}{12} = \$150.$$

If Drew borrowed $5000 from Carr for 6 months at the simple interest rate of 6% per annum and paid $2000 at the end of 2 months, how much should he pay at the end of 6 months in full settlement of his debt?

We may not consider that any part of the $2000 payment was for interest, for then Carr would be earning interest on interest, or compound interest. A solution of the problem can be obtained by considering that there are two separate transactions: (1) a loan by Carr to Drew of $5000 for 6 months; and (2) a loan by Drew to Carr of $2000 for 4 months.

At the end of 6 months, Drew owes Carr

$$\$5000 + 5000 \times .06 \times \tfrac{6}{12} = \$5150,$$

and Carr owes Drew

$$\$2000 + 2000 \times .06 \times \tfrac{4}{12} = \$2040.$$

Hence Drew should pay $5150 − 2040 = $3110.

Exercise 8

1. Find the simple interest on $400 for 8 months if the annual rate is: (a) 6%; (b) 5%; (c) $4\frac{1}{2}$%; (d) 4%; (e) $3\frac{1}{2}$%.

2. Henry borrowed $1000 for 8 months at the simple interest rate of 5% per annum. He paid $300 at the end of 3 months. How much should he pay at the end of the 8 months in full settlement of his debt?

3. If, in example 2, Henry paid $300 at the end of 3 months and $400 at the end of 5 months, how much should he pay at the end of 8 months?

4. If a loan of $800 is repaid at the end of 9 months by a payment of $836, find the rate of simple interest.

5. A loan of $900 made for 1 year at 4% is settled partly by a payment of $600 at the end of 6 months. If the debt is settled by a payment at the end of the year, find the total paid.

12. Ordinary and exact interest. The usual commercial procedure is to regard a year as consisting of 12 equal

months of 30 days each, thus assuming that there are 360 days in a year. This assumption simplifies interest calculations. The result is called *commercial* or *ordinary interest.* An interest calculation made by considering that there are 365 days in a year is called *exact interest.* Thus the interest on $1500 at 5% per annum is in either case $75 for 1 year. But for 90 days,

(a) the ordinary interest, $I = \$75 \times \frac{90}{360} = \$18.75$;

(b) the exact interest, $E = \$75 \times \frac{90}{365} = \$18.49$.

If we compare I with E by finding their ratio,

$$\frac{I}{E} = \frac{365}{360} = \frac{73}{72}, \text{ and } \frac{E}{I} = \frac{72}{73}.$$

That is, for the same principal, rate, and number of days, the ordinary interest is $\frac{73}{72}$ of the exact interest, and the exact interest is $\frac{72}{73}$ of the ordinary interest. Therefore, if the ordinary interest has been calculated, the exact interest may be found by deducting from it $\frac{1}{73}$ of the ordinary interest; if the exact interest has been calculated, the ordinary interest may be found by adding to it $\frac{1}{72}$ of the exact interest.

Stated briefly, the difference between I and E is $\frac{1}{73}$ of I or $\frac{1}{72}$ of E.

Thus, if the ordinary interest is calculated and found to be $18.75, the exact interest $= \$18.75 - \frac{1}{73}$ of $\$18.75 = \$18.49$.

If it is stated that the difference between the ordinary and the exact interest is 60 cents, we know at once that the ordinary interest is $73 \times 60¢ = \$43.80$, and the exact interest is $72 \times 60¢ = \$43.20$.

It is unusually simple to calculate ordinary interest if the rate is 6% per annum. Thus the ordinary interest on $5840 for 69 days at 6% is calculated as follows:

Interest for	60 days	=	$\$5840 \times .01$	=	\$58.40
" "	6 "	=	$\$58.40 \times .1$	=	5.84
" "	3 "	=	$\$5.84 \times \frac{1}{2}$	=	2.92
" "	69 "	=			\$67.16

If the rate had been 7%, then, since 7% is $\frac{7}{6}$ of 6%, the interest would have been $\$67.16 \times \frac{7}{6} = \$67.16 + \frac{1}{6} \times \$67.16 = \$78.35$. If the rate had been $4\frac{1}{2}\%$, which is $\frac{3}{4}$ of 6%, the interest would have been $\$67.16 \times \frac{3}{4} = \$67.16 - \frac{1}{4} \times \$67.16 = \$50.37$.

Exercise 9

1. In each of the following examples, find the ordinary and the exact interest separately, and check by using the conversion principle:

(a)	Interest on	\$7500	at 6%	per annum for	120 days
(b)	" "	\$7500	" 5%	" " "	150 "
(c)	" "	\$6837.50	" 7%	" " "	96 "
(d)	" "	\$5000	" $4\frac{1}{2}\%$	" " "	87 "
(e)	" "	\$6000	" $5\frac{1}{2}\%$	" " "	78 "

2. The difference between the ordinary and the exact interest on the same principal, for the same number of days, and at the same rate is \$5. Find the ordinary and the exact interest.

3. The ordinary interest on \$5000 for 100 days is \$75.
(a) Find the interest for a year.
(b) Find the annual rate of interest.

4. The ordinary interest on \$6000 at 6% per annum for an unknown period of time is \$100.
(a) How much would the interest be for 1 year?
(b) For what part of a year was the interest of \$100 paid?

5. The ordinary interest for 75 days at 6% per annum is \$225.
(a) How much would the interest be for 1 year?
(b) What was the principal?

6. Calculate the missing item in each of the following:

Principal	*Annual Rate*	*Time*	*Interest*
(a) \$4500	$5\frac{1}{2}\%$	60 days	. . . . ordinary
(b) \$6000	$4\frac{1}{2}\%$	80 days	. . . . exact

Principal	*Annual Rate*	*Time*	*Interest*
(c) $4000	....	90 days	$50 ordinary
(d) $4000	....	75 days	$50 exact
(e)	6%	70 days	$85 ordinary
(f)	5%	65 days	$80 exact
(g) $7000	7%	....	$92 ordinary
(h) $7500	4%	....	$120 exact

7. Real estate taxes in New York City are due May 31 and November 30. If the tax due is not paid on or before May 31, the city charges exact interest (ordinary time) at 7% per annum from May 1. Edward Collins has a tax bill of $26,325 due May 31, and on May 30 he borrows the money for 90 days at ordinary interest at $7\frac{1}{2}$%. Does Collins gain anything by borrowing $26,325? How high a rate of interest could Collins afford to pay for the loan without gain or loss?

13. Bank discount. Edward Post holds the following note as a result of a business transaction:

May 16, 1940.

Ninety days after date, I promise to pay to the order of Edward Post, $1860.00, at the Cannon National Bank, New York, with interest at 5%.

(Signed) JOHN COLE

Due, August 14, 1940.

On June 1, 1940, Post needs cash and asks for a loan from the Fleet Bank, where he has an account. Post endorses the note and turns it over to the Fleet Bank. The bank deducts a discount at 6% and credits his account $1860.02, called the *proceeds* of the note, as a result of the following calculation:

Interest at 5% on $1860 for 90 days: $\$1860 \times .05 \times \frac{90}{360} = \$23.25$;

maturity value of note: $\$1860 + \$23.25 = \$1883.25$;

time that the note still has to run: 74 days;

discount on $1883.25 for 74 days at 6%: $\$1883.25 \times .06 \times \frac{74}{360} = \$23.23$;

proceeds: $\$1883.25 - \$23.23 = \$1860.02$.

When the note falls due, August 14, 1940, the Fleet Bank will receive $1883.25 from John Cole and will cancel Post's liability on the loan. If Cole does not pay, the Fleet Bank will require Post to pay $1883.25, and Post will then try to collect from Cole.

The following points are to be noted:

(1) Post did not sell the note to the Fleet Bank.

(2) The Fleet Bank loaned Post $1883.25 and deducted a "discount at 6%" of $23.23, credited Post's account $1860.02, and held the note as evidence of the loan.

(3) The discount charge of $23.23 differs from interest in two respects: (a) It is deducted from the loan at the date that the loan is made, instead of being payable when the loan is repaid. (b) It is calculated on the maturity value of the note and not on the amount loaned.

(4) Since the Fleet Bank loaned $1860.02 and will receive $1883.25 at the end of 74 days, the interest earned by the bank in 1 year would be $\frac{\$23.23}{74} \times 360$, and the annual rate of interest earned would be $\frac{\$23.23}{74} \times 360 \times \frac{1}{\$1860.02}$ $= 6.07\%$, whereas the discount rate is 6%.

Exercise 10

1. Calculate the discount and the proceeds for each of the following notes:

	Note Dated	Discounted	Term	Face	Interest Rate	Discount Rate
(a)	Jan. 5	Feb. 10	90 days	$ 5000	5%	6%
(b)	Feb. 15	Feb. 15	60 "	2750	0	6%
(c)	March 1	April 10	90 "	7500	4%	6%
(d)	April 1	April 15	60 "	8000	6%	5%
(e)	April 15	April 15	120 "	15,000	0	5%

2. Find the difference in cost to the borrower of $50,000 for 90 days: (a) if he gives a note for $50,000 payable in 90 days with interest at 6%; or (b) if he gives a note for some amount payable in 90 days and the bank discounts the note at 6% so that the proceeds are $50,000.

3. A note for $6500 dated March 15 is due in 90 days with interest at 4% per annum.

(a) The note was discounted at 6% on April 20. Find the proceeds.

(b) The note was discounted at 6% and the proceeds were $6538.74. Find the date that the note was discounted.

(c) The note was discounted on May 10 and the proceeds were $6534. Find the rate of discount.

4. Find the rate of interest equivalent to a discount rate of:

(a) 5% per annum for 120 days;
(b) 6% " " " 60 " ;
(c) 4% " " " 90 " .

5. The discount on $5600 for 63 days is $48.62. Find the equivalent annual interest rate.

14. True discount. When a note for $5000 due at the end of 90 days is discounted by a bank at 6%, the bank discount is $75 and the proceeds are $4925. That is, at the end of 90 days the bank receives $5000 for a loan of $4925. The bank earns $75 on an investment of $4925 for 90 days, which is at an interest rate higher than 6% per annum.

The question "How much should I receive now so that at the end of 90 days a payment of $5000 will cancel the debt with ordinary interest at 6% per annum?" may be restated as: What principal will amount to $5000 at the end of 90 days with interest at 6%?

The unknown principal is now called the *true present worth* of $5000, and the difference between the true present worth and the final amount is called the *true discount.*

A principal of $1 at 6% a year will amount to $1.015 at the end of 90 days. In order that the amount shall be $5000 the principal should be $5000 ÷ 1.015, or $4926.11.

Hence the true present worth is $4926.11 and the true discount is $73.89, whereas the bank discount was $75.

But now 6% is not the rate of discount but the rate of interest, the interest on $4926.11 for 90 days at 6% being $73.89.

Exercise 11

1. Find the true present worth and the true discount:

(a) If $1000 is due at the end of 120 days and the annual interest rate is: 3%; 5%; 6%; 8%.

(b) If $2500 is due at the end of 80 days and the annual interest rate is: 6%; 4%; 3%; 2%.

2. The true discount is $65.80.

(a) Find the true present worth if the annual interest rate is 5% and the time is 90 days.

(b) Find the annual interest rate if the true present worth is $5000 and the time is 60 days.

(c) Find the time if the annual interest rate is 6% and the true present worth is $5000.

15. Equation of payments. In the settlement of mercantile accounts it is assumed that ordinary simple interest is to be charged for unpaid debts and credited for payments made in advance at 6% per annum unless another rate is specified.

The interest on \$500 for 17 days at 6% is $\$500 \times .06 \times \frac{17}{360}$. This may be written as $\$500 \times 17 \times .06 \times \frac{1}{360}$, which also indicates the interest on $\$500 \times 17$ for 1 day. That is, the interest on \$500 for 17 days is the same as the interest on $\$500 \times 17$ for 1 day.

Illustration

Porter bought merchandise from Quinn. On Quinn's books Porter is charged at dates when payments are due and credited with payments made, as follows:

Dr.		Cr.	
Jan. 15, 1940, due	$1000	Jan. 25, 1940, paid	$1000
Feb. 1, " "	5000	Feb. 10, " "	6000
May 15, " "	3000		

On what date should Porter pay the balance of $2000 without interest?

Select any date, preferably the last date in the account, May 15, 1940, as the focal date, the date on which our attention is focused. To this date, since 1940 was a leap year, Porter had the use of:

$1000	for	121	days, or	$121,000	for 1	day
$5000	"	104	" "	$520,000	" 1	"
$3000	"	0	" "	0	" 1	"
	A total of			$641,000	" 1	"

To the same date, Quinn had the use of:

$1000	for	111	days, or	$111,000	for 1	day
$6000	"	95	" "	570,000	" 1	"
	A total of			$681,000	" 1	"

To balance the advantages, Porter is entitled to the use of $40,000 for 1 day or to the use of the actual balance, $2000, for 20 days from May 15. Hence the equitable date of settlement is 20 days after May 15—that is, June 4, 1940.

If the account is settled after June 4, Porter should pay $2000 and interest. If the account is settled prior to June 4, Porter should pay $2000 less a discount.

Exercise 12

1. Find the equitable date of settlement in the illustrative problem by taking May 31 as the focal date.

2. Find the amount Porter should pay if he settles the account: (a) June 30; (b) May 20.

3. Brand borrowed $1000 for 1 year. He paid $275 at the end of 3, 6, 9, and 12 months. Show that he paid ordinary interest at 17% per annum (nearly).

4. If, in example 3, each of the payments had been $262.14, show that the annual rate of interest would have been 8% (nearly).

5. How much should each of 2 equal payments be if they are made at the end of 4 months and 12 months to repay a loan of $1000 made for a year with simple interest at 12% per annum?

6. H. Bond obtained a loan of $400 for 1 year. At the end of 3 months Bond pays $105; at the end of 6 months he pays $110; at the end of 9 months he pays $115; and at the end of the year he pays $120. Find the rate of interest Bond pays on the loan.

7. The loan of example 6 was not for a year but consisted of four items of $100 each, payable with interest at the end of 3, 6, 9, and 12 months. The payments Bond made were as in example 6. Find the rate of interest Bond paid.

8. On Walter Clark's books John Daly's ledger account appears as follows:

Dr.			Cr.		
Jan.	10, 1941, due	$ 500	Jan.	15, 1941, paid	$ 500
	20, " "	600	Feb.	1, " "	1600
Feb.	15, " "	1000	March	1, " "	2000
	20, " "	1000			
March	25, " "	2000			

(a) Find the date when Daly should pay the balance of $1000 without interest.

(b) The settlement is made March 25. How much should Daly pay if interest is at 6%?

(c) The settlement is made May 20. How much should Daly pay if interest is at 6%?

CHAPTER II

OPERATIONS WITH ALGEBRAIC NUMBERS

16. Algebraic numbers. All integral numbers (whole numbers) may be represented by points on a scale of equal divisions, each division point on the scale being marked for an integer. All common and decimal fractions may also

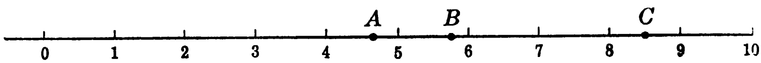


Fig. 1.

be indicated on this scale. Thus, in Fig. 1, $A = 4\frac{2}{3}$; $B = 5.8$; $C = 8.45$.

Addition and subtraction may be performed with the aid of the scale merely by counting. To find the value of $6 + 3$, start at 6 on the scale and count 3 toward the right. The mark numbered 9 is reached; therefore, the sum of 6 and 3 is 9.

$7 - 2$ means: How many steps must be taken from 2, the subtrahend, to reach 7, the minuend? Evidently we take 5 steps toward the right, the result being $7 - 2 = 5$.

$2 - 7$ means: How many steps must be taken from 7 to reach 2? The answer is again 5, but the motion in this case is toward the left, and we write $2 - 7 = -5$.

In order to represent such results, the scale is extended to the left of zero. The numbers on the scale to the left of zero are marked -1, -2, -3, and so on, to distinguish them from points to the right of zero which are marked $+1$, $+2$, $+3$, and so on. A number written without a sign before it is considered to be an arithmetic number preceded by a $+$ sign.

The complete scale, Fig. 2, extending from zero toward the right and toward the left is called the *algebraic scale of*

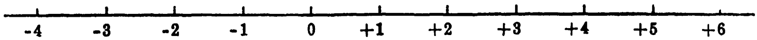


Fig. 2.

numbers. $+4$ means: $0 + 4$; start at 0 and count 4 toward the right. -3 means: $0 - 3$; start at 0 and count 3 toward the left.

$2 - 6$ would ordinarily mean that 6 is to be subtracted from 2, but the most that can be subtracted from 2 is 2. Of the 6 that were to be subtracted, there still remain 4 that are to be subtracted, indicated by -4, and we write $2 - 6 = -4$.

$2 - 6$ also means: How many steps must be taken from the mark 6 to reach the mark 2? Evidently 4 steps toward the left, indicated by -4, and $2 - 6 = -4$.

The signs $+$ and $-$, originally defined as signs that indicate addition and subtraction, are thus found to be equally well suited for indicating opposite directions.

$2 - 6$ also means $0 + 2 - 6$. Start at zero, count 2 toward the right, then 6 toward the left. The final position, 4 spaces to the left of zero, is indicated by -4.

The algebraic scale of numbers is also called the *scale of real numbers.* It contains every ordinary number whether it is integral or fractional, positive or negative, or zero. When dealing with real numbers, one number is said to be greater than another if its position on the scale is farther to the right. The sign $>$ is used to indicate "greater than," and the sign $<$ is used to indicate "less than." Thus

$$-7 > -10; \quad -5 < +1; \quad +2 < +10; \quad +1 > -3.$$

17. Algebraic addition. Algebraic addition is the process of combining a succession of $+$ and $-$ numbers into a single number. It may be accomplished by taking suc-

cessive steps on the algebraic scale, toward the right for + numbers and toward the left for − numbers. Thus $3 - 5 + 7 - 2$ means $0 + 3 - 5 + 7 - 2$. Start at zero, count 3 toward the right, then 5 toward the left, then 7 toward the right, and then 2 toward the left. The final point reached is 3 to the right of zero, and the algebraic sum is $+3$. The order of the numbers may be changed (see page 6), as: $3 - 5 + 7 - 2 = 3 + 7 - 5 - 2 = -5 - 2 + 3 + 7 = +3$. It is also convenient to regard plus numbers as gains and minus numbers as losses. The algebraic sum of the numbers is the result of combining the gains and the losses into a single number.

Exercise 13

1. Find the algebraic sum of each of the following:

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
6	6	6	6	−6	5	1.2	−1.23	−2	$-\frac{1}{2}$
7	7	7	−7	−7	−3	−2.3	−5.67	$\frac{2}{3}$	$\frac{1}{3}$
8	8	−8	−8	−8	7	4.7	2.34	$-\frac{1}{2}$	$\frac{1}{6}$
9	−9	−9	−9	−9	−8	−3.6	−0.46	$-\frac{1}{6}$	−1

2. Combine each of the following sets into a single number:

(a) $+ 3 - 1.874$

(b) $- 3 + 1.874$

(c) $- 5 + 2.63572$

(d) $1 - .68437$

(e) $.68437 - 1$

(f) $\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{6}$

(g) $- \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6}$

18. Algebraic subtraction. To subtract one algebraic number from another algebraic number, we must state the number of steps and the direction necessary to get from the point on the algebraic scale that indicates the subtrahend to the point that indicates the minuend. To subtract -6 from -2, it is necessary to take 4 steps toward the right. According to this definition of subtraction, the difference is $+4$, and $(-2) - (-6) = +4$.

We may reach -2 from -6 by going first from -6 to 0 and then from 0 to -2. The successive steps taken would be $+6$ and -2, which combine to make $+4$. That is, instead of moving from the subtrahend to the minuend directly, we may move from the subtrahend to zero and then to the minuend. Hence the rule: To subtract one algebraic number from another, reverse the sign of the subtrahend and proceed as in algebraic addition.

The correctness of an algebraic subtraction should be checked, as in arithmetic, by adding the difference to the subtrahend. The result should give the minuend.

Exercise 14

In each of the following examples, identify the minuend and the subtrahend. Then find the difference and check the result.

1.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
From:	10	10	10	10	-10	-10	-10	-10	-1.6	$-\frac{1}{3}$
Subtract:	4	12	-2	-12	2	12	-10	10	1.8	$\frac{1}{6}$

2. Subtract:

(a) -1 from $+1$

(b) $+1$ from -1

(c) $+2.783$ from -1.642

(d) -1.642 from $+2.783$

(e) $-2\frac{3}{8}$ from $-5\frac{1}{4}$

(f) $-\frac{3}{4}$ from $+\frac{1}{4}$

(g) $+\frac{3}{4}$ from $-\frac{1}{4}$

(h) $-\frac{3}{4}$ from $-\frac{1}{4}$

19. Algebraic addition and subtraction of groups. Instead of dealing with units, we may deal with groups of units. Then, if each group contains the same number of units, we may add or subtract the number of groups. Thus, if each group contains a dozen units and the letter d stands for 12, a combination such as $7d + 4d - 8d$ means that we are to combine 7 dozen + 4 dozen − 8 dozen. The word *dozen* now takes the place of the word *unit,* which was understood but not indicated in previous examples. The method of combining $7d + 4d - 8d$ is precisely the same

as the method of combining $7 + 4 - 8$. Now, however, the result is not 3 units but 3 groups of d each, or $3d$.

The combination $5a + 3b - 6a^2 + 4ab - 2a - 4a^2 - 4b$ means that we are to combine 5 groups of a each, $+3$ groups of b each, -6 groups of a^2 each, and so on. In this case, the number in each group is not the same, since a, a^2, b, and ab may, and generally do, represent different numbers. Some of the groups may, however, be combined. $5a - 2a$ combine to make $+3a$; $3b - 4b$ combine to make $-1b$, or $-b$; $-6a^2 - 4a^2$ combine to make $-10a^2$. The result of combining whenever it is possible to do so is $3a - b - 10a^2 + 4ab$.

Each of the parts, $5a$, $+ 3b$, $- 6a^2$, and so on, is called a *term*. A term is a number combination that is to be added algebraically to another number combination. In the example there were 7 terms in the original expression. The final result contains 4 terms. We say that an algebraic expression has been made simpler if it is expressed in fewer terms.

Like terms or *similar* terms are terms in which the same letters appear and the exponents attached to the letters in one are the same as those in the other. Thus $6a^3b^2$ and $5a^3b^2$ are like terms. Only like terms can be combined.

The number by which a letter or a letter combination is to be multiplied is called the *coefficient*. Thus, in $7ax^2$, the coefficient of ax^2 is 7; the exponent 2 applies to x only and not to ax. If no coefficient appears before a letter, the coefficient is understood to be 1. Thus, $ab = 1ab$.

An expression consisting of only one term is a *monomial;* one consisting of two terms is a *binomial;* one consisting of three terms is a *trinomial.* An expression consisting of more than one term is usually called a *polynomial.*

The algebraic subtraction of one polynomial from another is performed as follows:

Illustration

Subtract $6a^2 - 2ab + 2a - 3b$ from $5ab - 2a + 3a^2 + 3c$. Rewrite, placing each term of the subtrahend under a similar term of the minuend, if a similar term exists.

$$\begin{array}{ll} \text{Minuend:} & 5ab - 2a + 3a^2 + 3c \\ \text{Subtrahend:} & \underline{-2ab + 2a + 6a^2 \qquad - 3b} \\ \text{Difference:} & 7ab - 4a - 3a^2 + 3c + 3b \end{array}$$

Adding the difference to the subtrahend should give the minuend. Does it?

Exercise 15

Combine terms wherever possible; state how many terms there were in the example and how many there are in the answer:

1. $2x^2 - 5 + 3x - 7x^2 + 8 - 4x + 5x^2 - 3$
2. $a + 3b - 5a - 6b - 4b - 4a + 6a$
3. $a + 5ax + 7x - 8a - 3ax - ax$
4. $-3x - 4x^2 + 6x + x^3 - x + 2x^2$
5. $ab + 2ac - bc + 3ab - 4ac + 10bc$
6. $\frac{2}{3}a - \frac{1}{2}a + \frac{1}{4}a - \frac{1}{6}a$
7. $2.3x - 3.4x - 7.2x + x$
8. $3x^2 - 1.2x - 1.7x^2 + 1.2x$
9. $273a - 875a - 563a + 767a$
10. $689x^2 - 563x^2 - 197x^2 - 738x^2$
11. Subtract:

(a) $2a - 3b$ from $5a - 6b$.	(d) $2a - 3b$ from $2a + 3b$.
(b) $2a - 3b$ " $a - 6b$.	(e) $2a - 3b$ " $-2a - 3b$.
(c) $2a - 3b$ " $2x - 3y$.	(f) $2a - 3b$ " $-2a + 3b$.

12. If $A = 2a - 3b + 4c$, $B = 3a - 2b - c$, and $C = -a + b - 3c$, find the value of each of the following:

(a) $A + B + C$	(d) $B - A - C$
(b) $A + B - C$	(e) $C - A - B$
(c) $A - B - C$	(f) $B + C - A$

20. Parentheses. $5x + (6a - 3b) - (4y - 2z)$ means: Start with the group $5x$, add to it the group $6a - 3b$, and

then subtract the group $4y - 2z$. If the additions and the subtractions are performed, we obtain $5x + 6a - 3b - 4y + 2z$.

Upon comparing the result with the given form that contains parentheses, we note that:

(1) When a plus sign precedes the parentheses, the parentheses are removed without any changes in the signs of the terms that were within the parentheses, and the plus sign that preceded the parentheses disappears.

(2) When a minus sign precedes the parentheses, the parentheses are removed and, at the same time, the sign of every term that was within the parentheses is changed from plus to minus or from minus to plus, and the minus sign that preceded the parentheses disappears. Thus

$$5a - (-2x + 3a) + (-4a - 2x) = 5a + 2x - 3a - 4a - 2x = -2a.$$

Sometimes parentheses appear one within another. To avoid ambiguity, various forms are used, namely: [ ] *brackets*, $\{\quad\}$ *braces*, and $\overline{\qquad\qquad}$ *bar* or *vinculum*. Thus

$$5x - [3 - \{4 - \overline{x-2} + x\} + x] = 5x - [3 - \{4 - x + 2 + x\} + x]$$
$$= 5x - [3 - 4 + x - 2 - x + x] = 5x - 3 + 4 - x + 2 + x - x = 4x + 3.$$

Exercise 16

1. Remove parentheses and simplify the result in each case:

(a) $10 - (5 - 2x) + (7 - 3x)$
(b) $a - (2 - a - a^2) + (-a^2 - 2a + 3)$
(c) $(6x - 3y) - (5x - 2y) - (-3x + 2y)$
(d) $5x - [2x + (3 - \overline{4 - x})]$
(e) $8 + \{7 - [x + (3 - \overline{2x - 3})] - x\}$

2. State the operations of addition or subtraction to be performed in example 1; then perform the operations without using parentheses.

21. Multiplication. By the definition of exponents, page 3,

$$a^3 = a \cdot a \cdot a; \quad a^5 = a \cdot a \cdot a \cdot a \cdot a.$$

Then

$$a^3 \cdot a^5 = (a \cdot a \cdot a)\,(a \cdot a \cdot a \cdot a \cdot a) = a^8.$$

In multiplication, if the base is the same in two multipliers, the product has the same base, and the exponent is the sum of the exponents of the multipliers. Thus:

$$5^4 \times 5^6 = 5^{10}; \quad b^5 \cdot b = b^5 \cdot b^1 = b^6.$$

In general, if n and y are positive integers,

$$a^n \cdot a^y = a^{n+y},$$

The product of two monomials such as $2a^3b^2$ and $3ab^3c$ may be found in the following manner:

$$2a^3b^2 \cdot 3ab^3c = 2 \cdot 3 \cdot a^3 \cdot a^1 \cdot b^2 \cdot b^3 \cdot c = 6a^4b^5c.$$

Since in any product the order in which the factors appear may be changed at will, the letters are usually written in the order in which they appear in the alphabet.

Exercise 17

Simplify the products in the following:

1. $(3a^2) \times (4a^2)$
2. $(10ab) \times (6ac)$
3. $(abc) \times (a^2bx)$
4. $(2a^3b^2) \times (4a^4bc^2)$
5. $(\frac{3}{3}a^2x) \times (\frac{1}{2}ax^2)$
6. $3.75a^3x^2 \times 0.8a^2x$

22. Multiplication of polynomials. The signs + and − were defined originally as signs that indicate addition and subtraction. We then found that + 8 and − 3 have precise meanings with reference to the scale of algebraic numbers and that such numbers may be added together or one may be subtracted from the other. The question naturally arises: What shall we understand by such expressions as $(+8) \times (-3)$ and $(-7) \times (-4)$?

As these numbers stand, not connected with other numbers, the indicated multiplication conveys little or no meaning since multiplication in arithmetic is defined only

for positive numbers. But $(9 - 2) \times (8 - 3)$ certainly does have a meaning—namely, 7×5.

To formulate rules for multiplying $9 - 2$ by $8 - 3$ without combining $9 - 2$ into 7, and $8 - 3$ into 5, consider the following:

(1) If an article is worth 3 dollars and there are $7 - 2$ articles, the separate groups 7 and 2 are worth 7×3 and 2×3, and $7 - 2$ articles are worth $7 \times 3 - 2 \times 3$. But the value of $7 - 2$ articles is also $3 \times (7 - 2)$. Hence

$$3 \times (7-2) = 3 \times 7 - 3 \times 2.$$

In general, if a, c, and d are positive numbers and $c > d$,

$$a\,(c-d) = ac - ad.$$

(2) To find the value of $(a - b) \cdot (x - y)$ where a, b, x, and y are positive and $a > b$, $x > y$, replace $a - b$ by k. Then

$$(a - b) \cdot (x - y) = k\,(x - y) = kx - ky.$$

But

$$kx = x(a - b) = ax - bx,$$
$$ky = y(a - b) = ay - by;$$

and

$$kx - ky = (ax - bx) - (ay - by) = ax - bx - ay + by,$$

or

$$(a - b)\,(x - y) = ax - bx - ay + by.$$

The result of multiplying $a - b$ by $x - y$ is now summarized. These rules are applicable to the product of any two polynomials.

(a) Each term of the first factor, $+a$ and $-b$, is multiplied by each term of the second factor, $+x$ and $-y$, giving four terms in the product.

(b) The sign of a term in the product is $-$ when the signs of the two multipliers are different (one $+$ and the other $-$),

that is, when $+a$ is multiplied by $-y$ and when $-b$ is multiplied by $+x$.

(c) The sign of a term in the product is $+$ when the signs of the two multipliers are the same (both $+$ or both $-$), that is, when $+a$ is multiplied by $+x$ and when $-b$ is multiplied by $-y$.

(d) The terms of the product are added algebraically.

Illustration

Multiply:	$5x^3 - 3x^2 - 8x + 4$
by:	$2x^2 - 4x - 3$
Multiply by $2x^2$:	$+10x^5 - 6x^4 - 16x^3 + 8x^2$
" " $-4x$:	$-20x^4 + 12x^3 + 32x^2 - 16x$
" " -3:	$-15x^3 + 9x^2 + 24x - 12$
Product:	$+10x^5 - 26x^4 - 19x^3 + 49x^2 + 8x - 12$

The proof of correctness is established as follows: Let x be given any numerical value whatever, say $x = 3$. The first factor reduces to $+88$, the second factor reduces to $+3$, and the product reduces to $+264$. Since $(+88) \times (+3) = +264$, the algebraic multiplication is probably correct.

Note that when 7843 and 956 are multiplied, the rules (a) and (d) of algebraic multiplication are obeyed. There are actually 12 multiplications made, and the results are added together.

Exercise 18

Perform the following multiplications, and check each result by letting $x = -\frac{2}{3}$:

1. $(2x - 3) \cdot (3x - 4)$
2. $(2x^2 - 3x - 4) \cdot (3x - 2)$
3. $(2x^2 - 3x - 4) \cdot (x^2 - 2x + 3)$
4. $(x^2 + x + 1) \cdot (x - 1)$
5. $(x^2 - x + 1) \cdot (x + 1)$
6. $(x^2 - x + 1) \cdot (x^2 + x + 1)$
7. $(x-1)(x-2)(x-3)(x-4)$
8. $(x^2 - 3x - 6)^2 - (x^2 + 3x - 6)^2$
9. Why may we not use $x = 1$ or $x = 0$ to check multiplications?

10. Show that the following results are true and state each in words (see also page 6):

(a) $(a+b)^2 = a^2 + 2ab + b^2$

(b) $(a-b)^2 = a^2 - 2ab + b^2$

(c) $(a+b)(a-b) = a^2 - b^2$

11. Use the rules of example 10 to write the following products at sight:

(a) $(x+3)^2$

(b) $(x-1)^2$

(c) $(x-5)(x+5)$

(d) $(2x+3)^2$

(e) $(2x-3)^2$

(f) $(2x+3)(2x-3)$

(g) $6\frac{2}{3} \times 7\frac{1}{3}$

(h) 59×61

(i) 93×87

(j) $6\frac{1}{3} \times 6\frac{1}{3}$

(k) 6.03×5.97

(l) 147×147

23. Division of algebraic numbers. $a \div b = c$, or $\frac{a}{b} = c$, means that some number c must be found such that $c \times b$ shall give a. That is, $a^6 \div a^2$ means that some number must be found so that when it is multiplied by a^2 the product shall be a^6. Since $a^4 \times a^2 = a^6$, $a^6 \div a^2 = a^4$.

In general, for positive integral values of m and n, $a^m \cdot a^n = a^{m+n}$, and if $m > n$, $a^m \div a^n = a^{m-n}$.

The student may prove that the rules for signs in division are the same as in multiplication—namely, like signs give $+$, unlike signs give $-$.

If a polynomial is to be divided by a monomial, each term of the polynomial is divided by the monomial. Thus

$$\frac{16a^3b - 12a^4b^2 - 10a^5b^3}{-2a^3b} = -8 + 6ab + 5a^2b^2.$$

The proof of the correctness of the result is

$$-2a^3b\,(-8 + 6ab + 5a^2b^2) = +16a^3b - 12a^4b^2 - 10a^5b^3.$$

Exercise 19

Simplify each of the following, and check the results:

1. $10a^4 \div 2a$
2. $18a^4b^2 \div 3ab^2$
3. $18x^4 \div -3x^2$
4. $(30x^5 - 25x^4 - 20x^3) \div -5x^2$
5. $(8a^3b - 48a^2b^3) \div -8a^2b$
6. $(x^4y - x^3y) \div x^3y$
7. $(8a^3b - 16a^2b^3) \div 4ab$
8. $(6x^3y^3 - 2x^2y^2 - 4x^2y^3) \div 2xy$
9. $(x^4 - x) \div x$
10. $(ax + bx - cx) \div x$

24. Successive approximations. Addition and multiplication are direct processes. Subtraction and division are indirect processes, the correctness of the results being proved by addition and multiplication respectively. To square or to cube a number requires merely a knowledge of multiplication. The reverse processes, square root and cube root, are indirect. As a general rule, an indirect operation is performed by making an approximation, testing the result, making a better approximation, testing the new result, and so on.

Long division in arithmetic is performed by using successive approximations. Thus, to divide 73,425 by 236, we take the following steps:

1st approximation: 734 hundreds $\div$ 236 gives approximately 3 hundreds, or 300.

Test: $300 \times 236 = 70{,}800$. There still remains $73{,}425 - 70{,}800$, or 2625, to be divided by 236.

2nd approximation: 262 tens $\div$ 236 gives approximately 1 ten, or 10, and the second approximation is $300 + 10$, or 310.

Test: $10 \times 236 = 2360$. There still remains $2625 - 2360$, or 265, to be divided by 236.

3rd approximation: 265 units $\div$ 236 gives approximately 1 unit, and the third approximation is $300 + 10 + 1$, or 311.

Test: $1 \times 236 = 236$. There still remains 29 to be divided by 236.

4th approximation: $29 \div 236$, or 290 tenths $\div$ 236, gives approximately 1 tenth, or 0.1, and the fourth approximation is 311.1, the remainder being 5.4.

The ordinary setup is an abbreviation of the process given in detail, and appears as illustrated at the right. The proof of correctness is that $311.1 \times 236 + 5.4 = 73{,}425$.

$$\begin{array}{r|l}
 & 311.1 \\ \hline
236 & 73425.0 \\
 & \underline{708} \\
 & 262 \\
 & \underline{236} \\
 & 265 \\
 & \underline{236} \\
 & 290 \\
 & \underline{236} \\
 & 54
\end{array}$$

25. Division by polynomials. Algebraic division by a monomial presents no difficulty. But when the divisor consists of more than one term, we resort to the method of successive approximations very much as we do in long division in arithmetic.

In the case of numbers, the position of a digit determines its value — that is, 236 means 2 hundreds + 3 tens + 6 units.

The terms of an algebraic expression are arranged in some order, and that order is retained until the problem is completed. The value of x^3 is greater than that of x if x exceeds 1, and is less than that of x if x is between 0 and 1. Hence we may arrange an algebraic expression containing x by having the exponents appear either in a decreasing order or in an increasing order.

Illustration

$$(22x - 7x^3 - 6 + 6x^4 - 15x^2) \div (3 + 2x^2 - 5x)$$

The terms are arranged in decreasing powers of x.

$$\begin{array}{r|rrrrr}
 & 3x^2 & +4x & -2 & & \\ \hline
2x^2 - 5x + 3 & 6x^4 & -7x^3 & -15x^2 & +22x & -6 \\
 & 6x^4 & -15x^3 & +9x^2 & & \\ \cline{2-4}
 & & +8x^3 & -24x^2 & & \\
 & & +8x^3 & -20x^2 & +12x & \\ \cline{3-5}
 & & & -4x^2 & +10x & \\
 & & & -4x^2 & +10x & -6 \\ \cline{4-6}
 & & & & & 0
\end{array}$$

1st approximation: $6x^4 \div 2x^2 = 3x^2$.

Test: $3x^2 \cdot (2x^2 - 5x + 3) = 6x^4 - 15x^3 + 9x^2$. There still remains $8x^3 - 24x^2 + 22x - 6$ to be divided.

2nd approximation: $8x^3 \div 2x^2 = 4x$.

Test: $4x \cdot (2x^2 - 5x + 3) = 8x^3 - 20x^2 + 12x$. The second approximation is $3x^2 + 4x$, and there still remains $-4x^2 + 10x - 6$ to be divided.

3rd approximation: $-4x^2 \div 2x^2 = -2$.

Test: $-2 \cdot (2x^2 - 5x + 3) = -4x^2 + 10x - 6$. The third approximation is $3x^2 + 4x - 2$ and there is no remainder. The exact quotient is $3x^2 + 4x - 2$.

To prove the correctness of any division, use the relation, dividend = divisor $\times$ quotient + remainder.

Exercise 20

Perform the divisions indicated in the following examples and check by letting $x = -3$.

1. $(9x^2 - 4) \div (3x - 2)$

2. $(9x^2 - 4) \div (3x + 2)$

3. $(27x^3 - 8) \div (3x - 2)$

4. $(27x^3 - 8) \div (3x + 2)$

5. $(27x^3 + 8) \div (3x + 2)$

6. $(27x^3 + 8) \div (3x - 2)$

7. $(4x^4 - x^2 + 4) \div (2x^2 - 3x + 2)$

8. $(x^4 + x^3 - 3x^2 - x + 2) \div (x^2 + x - 2)$

9. $(x^4 + x^3 - 3x^2 - x + 2) \div (x^2 - 2x + 1)$

10. $(x^4 + x^3 - 3x^2 - x + 2) \div (x^2 + 3x + 2)$

26. Casting out nines. Arithmetic computations are usually checked by casting out the 9's in each number—that is, by using the remainder, called the *check number,* which is found after dividing the number by 9.

To find the check number of 8573, note that the number is 8 thousands + 5 hundreds + 7 tens + 3 units. When 1, 10, 100, and 1000 are divided by 9, the remainder in each case is 1. Therefore the remainder for any figure in any place is the same as the figure in that place. Hence when 8573 is divided by 9, the separate remainders are 8,

5, 7, and 3. The total remainder is $8 + 5 + 7 + 3$, or 23, which in turn gives the remainder $2 + 3$, or 5, the check number of 8573. That is, the check number when 9's are cast out is found by adding the digits together.

Let M be any number written as a sequence of digits, the decimal point being disregarded. Then if M is divided by 9, there will be an integral quotient, a, and an integral remainder, m. Hence we may write $M = 9a + m$, where m is the check number of M. Similarly $N = 9b + n$, where n is the check number of N.

$$M \times N = (9a + m)(9b + n) = 9^2ab + 9an + 9bm + mn.$$

Hence if $M \times N$ is divided by 9, the quotient is $9ab + an + bm$, and the remainder, the check number, is mn. That is, the check number of the product of two numbers is the same as the product of the check numbers of the factors. Thus

$$673 \times 587 = 395{,}051.$$

The check numbers of the given factors are 7 and 2, and the check number of the product is 5. But 7×2, or 14, gives the check number 5, and we conclude that the product is probably correct.

A division may be checked by finding the check numbers of the dividend, the divisor, the quotient, and the remainder, and then testing whether the four check numbers obey the relation: quotient $\times$ divisor $+$ remainder $=$ dividend.

Thus $78{,}432 \div 467$ gives the quotient 167 and the remainder 443. The check numbers are: dividend, 6; divisor, 8; quotient, 5; remainder, 2. Hence $8 \times 5 + 2$, or 42, should give the same check number as the dividend, 6. Since this is so, the division is probably correct.

Notice that reversing the order of two figures leaves the check number unchanged. Hence the check by casting out 9's cannot uncover such an error.

27. Casting out elevens. When 42 is divided by 9, the quotient is 4 and the remainder is 6; but we may also say that the quotient is 5 and the remainder is -3. When 10 is divided by 11, we may say that the quotient is 1 and the remainder is -1. That is, the remainders when 1, 10, 100, 1000, and so on, are divided by 11 are $+1$, -1, $+1$, -1, and so forth. Therefore the remainder for any figure in any place is the same as the figure in that place, but the signs of the separate remainders are alternately $+$ and $-$, the sign for the figure in unit's place being $+$. Hence, when 25,734 is divided by 11, the remainder is $+4-3+7-5+2$, or 5, which is the check number of 25,734.

The check number of 8573 is $+3-7+5-8$, or -7. The remainder, -7, was the result of making the quotient too large by 1. Diminishing the quotient by 1 increases the remainder by 11, and the check number therefore is also $+4$. That is, when 11's are cast out, the check numbers -1, -2, -3, and so on, are the same as the check numbers $+10$, $+9$, $+8$, and so on.

The rules for checking computations by casting out 11's are the same as those used when casting out 9's. An error of inversion of two adjacent digits not uncovered by casting out 9's is uncovered when 11's are cast out.

It is possible to formulate rules for finding the remainder for divisors other than 9 or 11, but the rules are too complicated to be of much practical use.

Exercise 21

1. Find the check numbers by casting out 9's and also 11's of:

(a) 2345 (b) 5432 (c) 38,412

2. Perform the multiplications and the divisions indicated and check by casting out 9's and also 11's:

(a) 376×437 (d) $6.7432 \div 1.23$
(b) $67{,}432 \div 123$ (e) $880 \div .125$
(c) $3.76 \times .437$ (f) $880 \times .125$

3. Prove that for any divisor, a, the check number of the sum of two numbers M and N is the sum of the check numbers of M and N.

28. Factoring numbers. An integral number is factored when it is expressed as the product of two or more smaller integral numbers. A number that cannot be factored is a prime number. Thus 1, 2, 3, 5, 7, 11, and 13 are prime numbers. If one factor of a number is discovered, another factor is the quotient obtained by dividing the given number by the known factor. Each factor may be factored in turn until the factors are prime numbers.

Any number M may be written in the forms:

(a) $M = 10a + b$, where b is the last figure of M;

(b) $M = 100c + d$, where d is the number shown by the last two figures of M;

(c) $M = 9e + f$, where f is the check number of M when 9's are cast out;

(d) $M = 11g + h$, where h is the check number of M when 11's are cast out.

Of course, there are other ways of expressing M, but these four forms enable us to recognize certain factors at sight.

(1) Since in (a) $10a$ is divisible by 2 and by 5, M is divisible by 2 or by 5 if b, the last figure, is divisible by 2 or by 5.

(2) Since in (b) $100c$ is divisible by 4 and by 25, M is divisible by 4 or by 25 if d, the number shown by the last 2 figures, is divisible by 4 or by 25.

(3) Since in (c) $9e$ is divisible by 3 and by 9, M is divisible by 3 or by 9 if f, the check number of M when 9's are cast out, is divisible by 3 or by 9.

(4) Since in (d) $11g$ is divisible by 11, M is divisible by 11 if h, the check number when 11's are cast out, is 0 or is divisible by 11.

That is, we can easily see whether 2, 3, 4, 5, 9, 25, or 11 is a factor, and, if it is, we can find another factor.

Illustration

Factor $M = 84{,}416{,}013$.

The check number when 9's are cast out is 27, which is divisible by 9. Hence 9 is a factor.

The check number when 11's are cast out is -11 or 0. Hence 11 is a factor.

$$\begin{aligned} M &= 9 \times 9{,}379{,}557 = 9 \times 9 \times 1{,}042{,}173 = 9^3 \times 115{,}797 \\ &= 9^3 \times 3 \times 38{,}599 = 9^3 \times 3 \times 11 \times 3509 = 9^3 \times 3 \times 11^2 \times 319 \\ &= 9^3 \cdot 3 \cdot 11^2 \cdot 11 \cdot 29 = 3^7 \cdot 11^3 \cdot 29 \end{aligned}$$

Exercise 22

Factor the following numbers:

1. 16,335	**6.** 27,588
2. 19,305	**7.** 172,425
3. 14,553	**8.** 129,415
4. 155,727	**9.** 114,345
5. 1,029,105	**10.** 47,544,651

29. Uses of factoring numbers. In calculations with fractions, it is always desirable that: (a) a fraction shall be expressed with small numbers if possible—that is, it shall be reduced to lowest terms; (b) the product, the quotient, or the sum shall be obtained as rapidly as possible. Factoring is helpful in all such cases.

Illustrations

1. Reduce $\frac{693}{847}$ to lowest terms.

When numerator and denominator are factored, the fraction becomes

$$\frac{693}{847} = \frac{11 \times 3^2 \times 7}{11^2 \times 7} = \frac{11 \times 3^2 \times 7}{11 \times 11 \times 7} = \frac{9}{11}.$$

The 11's and the 7's are canceled, and the simplified result is $\frac{9}{11}$.

2. $$\frac{13}{693} + \frac{9}{847} = \frac{13}{11 \times 3^2 \times 7} + \frac{9}{11^2 \times 7}.$$

The least common denominator is $11^2 \times 3^2 \times 7$, because this is the smallest integral number to which both the given denominators may be changed by multiplication. Then, since the numerator and the de-

nominator may be multiplied by the same number without changing the value of a fraction, the fractions become

$$\frac{13 \times 11}{11^2 \times 3^2 \times 7} + \frac{9 \times 3^2}{11^2 \times 3^2 \times 7}$$

The sum is

$$\frac{13 \times 11 + 9 \times 9}{11^2 \times 3^2 \times 7} = \frac{224}{11^2 \times 3^2 \times 7} = \frac{2^5 \times 7}{11^2 \times 3^2 \times 7} =$$

$$\frac{2^5}{11^2 \times 3^2} = \frac{32}{1089}.$$

Exercise 23

1. Reduce the following fractions to lowest terms:

(a) $\dfrac{3^2 \times 5^3 \times 7 \times 11}{3^3 \times 5 \times 7^2 \times 11}$ (d) $\dfrac{572}{14157}$

(b) $\dfrac{561}{765}$ (e) $\dfrac{1551}{4653}$

(c) $\dfrac{5049}{8415}$ (f) $\dfrac{17061}{18612}$

2. Simplify the following by performing the indicated operations, and express the result as simply as possible.

(a) $\dfrac{7}{108} + \dfrac{5}{72}$ (d) $\dfrac{3}{5} \times \dfrac{6}{35} \div \dfrac{81}{100}$

(b) $\dfrac{65}{231} - \dfrac{55}{2695}$ (e) $\dfrac{87351}{96162} \times \dfrac{21855}{23823}$

(c) $\dfrac{5}{156} - \dfrac{15}{585}$ (f) $\dfrac{8}{429} - \dfrac{16}{6435}$

30. Factors of algebraic expressions. The rules for operating with algebraic fractions are the same as those for arithmetic fractions. It is therefore necessary to factor algebraic expressions. A number that was obtained by multiplying two prime numbers is factorable, but it may be very difficult to find the factors. This is also true of algebraic expressions. However, some algebraic products may be written at sight, and the form of the product often enables us to write the factors.

The following are basic forms:

(1) $a(x - y + 1) = ax - ay + a$. A factor of each term, a, is also a factor of the entire expression. Another factor is the quotient obtained by dividing the expression by the known factor, a.

Thus $$5x^3 - 35x^2 + 5x = 5x(x^2 - 7x + 1).$$

(2) $(a + x)(a - x) = a^2 - x^2$. The difference between the squares of two expressions arises from the product of a sum and a difference. Thus $4x^2 - 9$ is the difference between the squares of $2x$ and 3. Hence

$$4x^2 - 9 = (2x + 3)(2x - 3).$$

Similarly, $(x + a)^2 - (y - b)^2$ is the difference between the squares of $x + a$ and $y - b$. Hence

$$(x + a)^2 - (y - b)^2 = [(x + a) + (y - b)][(x + a) - (y - b)].$$

(3) $$(a + x)^2 = (a + x)(a + x) = a^2 + 2ax + x^2,$$
and $$(a - x)^2 = a^2 - 2ax + x^2.$$

A trinomial reduces to the square of a binomial if two terms of the trinomial are squares and the third term is + or − twice the product of the square roots of the other terms.

Thus $4x^2 - 12x + 9$ is a trinomial in which $4x^2$ is the square of $2x$, 9 is the square of 3, and the remaining term, $-12x$, is twice the product of $2x$ and 3 if different signs are given to $2x$ and 3. Hence

$$4x^2 - 12x + 9 = (2x - 3)^2 \text{ or } (3 - 2x)^2.$$

Similarly, in $9x^4 + 30x^2y^3 + 25y^6$, $9x^4$ is the square of $3x^2$, $25y^6$ is the square of $5y^3$, and $+30x^2y^3$ is twice the product of $3x^2$ and $5y^3$ if like signs are given to $3x^2$ and $5y^3$. Hence $9x^4 + 30x^2y^3 + 25y^6 = (3x^2 + 5y^3)^2$ or $(-3x^2 - 5y^3)^2$.

Exercise 24

1. Verify each of the following by multiplication:

(a) $x^2 - 10x + 25 = (x - 5)^2$
(b) $4x^2 + 20x + 25 = (2x + 5)^2$
(c) $9x^2 - 25y^2 = (3x + 5y)(3x - 5y)$
(d) $15x^4 - 10x^3 + 5x^2 = 5x^2(3x^2 - 2x + 1)$

(e) $(x + 1)^2 - 9^2 = (x + 1 + 9)(x + 1 - 9)$
$= (x + 10)(x - 8)$

(f) $4 - (2x - 3)^2 = [2 + (2x - 3)][2 - (2x - 3)]$
$= (2x - 1)(5 - 2x)$

(g) $9x^4 - 4x^2 = x^2(9x^2 - 4) = x^2(3x + 2)(3x - 2)$

(h) $5x^4 - 5 = 5(x^4 - 1) = 5(x^2 + 1)(x^2 - 1)$
$= 5(x^2 + 1)(x + 1)(x - 1)$

(i) $x^2 + 6x + 8 = x^2 + 6x + 9 - 1 = (x + 3)^2 - 1 = (x + 2)(x + 4)$

(j) $x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2 = (x^2 + 1)^2 - x^2$
$= (x^2 + x + 1)(x^2 - x + 1)$

2. Factor the following:

(a) $45x^4 - 125x^2$
(b) $144x^5 - 480x^4 + 400x^3$
(c) $7x^2 - 28$
(d) $3x^2 - 12x + 12$
(e) $9ax^4 - 9a$
(f) $5ax^5 + 30ax^4 + 45ax^3$

3. Simplify the combinations:

(a) $\dfrac{3a^2}{2x^3} \cdot \dfrac{8x^5}{51a^4} \div \dfrac{2x}{17a}$

(b) $\dfrac{2x - ax}{10a + 2x} \cdot \dfrac{35ax + 7x^2}{28a - 14a^2}$

(c) $\dfrac{a^2 - b^2}{a^2 + ab} \cdot \dfrac{3x - 15}{ab - b^2} \div \dfrac{5x^2 - 25x}{6abx}$

(d) $\dfrac{x^2 + 6x + 9}{x^2 - 4x + 4} \div \dfrac{x^2 - 9}{x^2 - 4} \cdot \dfrac{x - 2}{x + 2}$

(e) $\dfrac{2}{x^2 + x} + \dfrac{3}{x^2 - x}$

(f) $\left(1 + \dfrac{2ax}{a^2 + x^2}\right) \div \left(1 - \dfrac{2ax}{a^2 + x^2}\right) \cdot \dfrac{3a - 3x}{5ax + 5x^2}$

31. Factor theorem. The principle used in casting out 9's or 11's is part of a general relation that applies to algebraic expressions. If P is a polynomial containing powers of x, and the $(x - a)$'s are cast out of P by dividing P by $x - a$, there is a remainder, R, which does not contain x, and a quotient, Q, which may or may not contain x. Hence we may write

$$P = (x - a)Q + R.$$

Since this relation is true for all values of x, we may set $x = +a$. We then find that by substituting $+a$ for x in P, the result is R. Thus when $P = x^3 + 3x^2 - 4$ is divided by $x - 2$, the remainder, R, is found by substituting $+2$ for x in P. Hence $R = 2^3 + 3 \cdot 2^2 - 4 = +16$. When P is divided by $x + 3$, the remainder, R, is found by substituting -3 for x in P. Hence, $R = (-3)^3 + 3(-3)^2 - 4 = -4$. That is, R, the check number of P when $(x - a)$'s are cast out, can be found without performing the division.

In particular, if $P = a + bx + cx^2 + dx^3 + \ldots$ is divided by $x - 1$, $R = a + b + c + d + \ldots$, and if P is divided by $x + 1$, $R = a - b + c - d + \ldots$

Now if $x = 10$ and $a, b, c, d, \ldots$ are positive integers from 0 to 9, P is the algebraic form of a number in which the unit's, ten's, hundred's, etc. digits are $a, b, c, \ldots$ Division by $x - 1$ and $x + 1$ are division by 9 and 11 respectively, and the results, $R = a + b + c + d + \ldots$ for 9 and $R = a - b + c - d + \ldots$ for 11, are the rules previously given for casting out 9's and 11's.

If, when $(x - a)$'s are cast out of P, the remainder, R, is zero, $x - a$ is a factor of P, and another factor is the quotient found by dividing P by $x - a$.

Illustration

Factor $P = x^4 - 5x^3 - 7x^2 + 29x + 30$.

Try the divisors $x + 1$, $x - 1$, $x + 2$, $x - 2$, etc.

For $x + 1$, $R = (-1)^4 - 5(-1)^3 - 7(-1)^2 + 29(-1) + 30 = 0$.

For $x + 2$, $R = (-2)^4 - 5(-2)^3 - 7(-2)^2 + 29(-2) + 30 = 0$.

For $x - 3$, $R = (3)^4 - 5(3)^3 - 7(3)^2 + 29(3) + 30 = 0$.

Hence $x + 1$, $x + 2$, $x - 3$, are factors, and the remaining factor, $x - 5$, may be found by dividing P by $(x + 1)(x + 2)(x - 3)$.

Hence $x^4 - 5x^3 - 7x^2 + 29x + 30 = (x + 1)(x + 2)(x - 3)(x - 5)$.

Exercise 25

Factor each of the following and check by multiplication:

1. (a) $x^2 - 9$; (b) $x^2 - 4x + 4$; (c) $x^2 + 6x + 9$
2. (a) $x^2 - 3x + 2$; (b) $x^2 + 3x + 2$; (c) $x^2 - x - 2$; (d) $x^2 + x - 2$
3. (a) $x^3 + 1$; (b) $x^3 - 8$; (c) $x^3 + 27$; (d) $x^3 - 64$
4. $x^3 - 9x^2 + 26x - 24$
5. $x^4 + 2x^3 - 4x^2 - 5x - 6$

Simplify the following:

6. $\dfrac{x^2 - 5x + 6}{x^2 - 4}$
7. $\dfrac{x^2 - x - 2}{x^2 + x - 2} \div \dfrac{x^2 + 2x + 1}{x^2 + 4x + 4}$
8. $\dfrac{x + 2}{x^3 - 8} - \dfrac{1}{x^2 - 4}$
9. $\dfrac{1}{x^2 + 5x + 6} + \dfrac{1}{x^2 + 7x + 12}$
10. $\dfrac{1}{x^2 + 5x - 6} \div \dfrac{1}{x^2 + 2x - 3}$

CHAPTER III

EQUATIONS

32. Equation and identity. The statement that one number or one algebraic expression is equal to another, is an *equality* which may be an *identity* or an *equation*.

The equality $(x + 3)^2 = x^2 + 6x + 9$ is an identity because it merely expresses the same algebraic expression in two different forms. The identity is true for any value that may be assigned to x. The equality $x^2 + 6 = 5x$ becomes an identity for $x = 2$ and for $x = 3$ but for no other values of x. The equality is called an *equation*, and the values of x that make the equation an identity are called the *roots of the equation*. An equality that reduces to an identity for one or more values of the letters, but not for all values, is an equation. The terms to the left of the equal sign constitute the *first member* and the terms to the right of the equal sign constitute the *second member* of the equation.

When all the indicated algebraic operations in an equality are performed and the two members reduce to the same expression, the equality is an identity; if they do not, it is an equation.

Since the two members of an equation represent the same number, the equality is not destroyed if the same mathematical operation is performed upon both members. The process of solving an equation consists of a series of changes in the form of the equation until a form is found in which the roots are evident.

In any equality such as $a - b + c = d$, subtracting $-b + c$ from each member gives $a = d + b - c$.

It is to be noted that the terms $-b$ and $+c$ disappear from the first member and reappear in the second member as $+b$ and $-c$. We conclude therefore that a term may be removed from one member and placed in the other member if at the same time its sign is changed. We say then that the term has been *transposed.*

Illustrations

1. Solve $5a - 2 = 2a + 10$ for a.

Transpose -2 and $+2a$,	$5a - 2a = +10 + 2.$
Combine terms,	$3a = 12.$
Divide each member by 3,	$a = 4.$

Check: Substitute in the given equation 4 for a. The result, $5 \times 4 - 2 = 2 \times 4 + 10$, or $18 = 18$, is an identity, and the root of the equation is 4.

2. Solve $\dfrac{3}{x-2} = \dfrac{2}{x-3} + \dfrac{1}{x-1}$ for x.

Multiply both members of the equation by $(x - 2)\ (x - 3)\ (x - 1)$. The equation then appears without fractions, and we say that the equation has been *cleared of fractions.* The new form is

$$3(x - 3)\ (x - 1) = 2(x - 2)\ (x - 1) + 1(x - 2)\ (x - 3).$$

After the indicated multiplications are performed, the result is

$$3x^2 - 12x + 9 = 2x^2 - 6x + 4 + x^2 - 5x + 6.$$

Transpose and combine terms, and find the new form

$$-x = +1, \text{ or } x = -1.$$

The check is left as an exercise for the student.

3. Solve	$x^2 + 6 = 5x$ for x.
Transpose:	$x^2 - 5x + 6 = 0.$
Factor:	$(x - 2)\ (x - 3) = 0.$

The product of two or more factors can be 0 only if one of the factors is 0. Since the product of $x - 2$ and $x - 3$ is 0, it must be because either $x - 2 = 0$ or $x - 3 = 0$. Hence $x = +2$ or $x = +3$. Each of these values of x reduces the given equation to an identity, and the roots of $x^2 + 6 = 5x$ are 2 and 3.

Exercise 26

Solve the following equations and check the roots:

1. $2x - 5 = x + 2$
2. $5x - 8 = 2x + 4$
3. $3x - (x - 2) = 5 - (2x - 3)$
4. $6 - 3x = x - 1$
5. $2(x - 2)(3x + 2) = (6x - 1)(x - 3) + 22$
6. $1.32x - 2.45 = 2.19 - x$
7. $2a - 7 = 5 - (3a - 2)$
8. $2a(a - 3) = a(2a - 3) - 6$
9. $x(x + 1) - (x - 1)(x - 2) = 6$
10. $(a + 2)(a - 3) - (a - 2)(a + 3) = -10$
11. $\frac{2}{3}x - \frac{1}{2} = x - 1\frac{1}{3}$
12. $\frac{1}{2}x + \frac{2}{3} = x$
13. $x - \frac{1}{3} = \frac{1}{6} - x$
14. $\frac{1}{2}x - 1 = 1 - \frac{1}{3}x$
15. $2\frac{1}{2} - \frac{1}{3}x = \frac{1}{3}x - \frac{1}{2}$
16. $\dfrac{2}{x} = 3 + \dfrac{5}{x}$
17. $\dfrac{x}{2} + \dfrac{1}{3} = \dfrac{x}{5}$
18. $\dfrac{2}{3x} + \dfrac{1}{2} = \dfrac{5}{4x}$
19. $\dfrac{3}{x - 2} = \dfrac{2}{x - 3} + \dfrac{1}{x - 1}$
20. $x^3 + 2 = 2x^2 + x$
21. $x^3 + 9x^2 + 26x + 24 = 0$

More than one letter appears in each of the following equations. When solving for any one letter, treat the remaining letters as if they were known numbers.

22. $ax + 5 = 3x + a$ — for x; for a.
23. $ah + 3a = 5h - 2$ — for a; for h.
24. $xy + 3 = 2x - 3y$ — for x; for y.
25. $5xy - 5x = by + b$ — for x; for y; for b.
26. $p + prt = s$ — for p; for r; for t.
27. $m - mrt = p$ — for m; for r; for t.

33. Algebraic method. The algebraic method of solving problems may be described as follows: Assume that the number sought is known, and represent it by a letter. Use the letter just as if the number that it represents were known, and proceed to prove that it is correct just as you would if you had guessed the answer. The proof is indicated by using the appropriate signs of addition, subtraction, and so on, and the form that results from the indi-

cated proof is an equation. The solution of the equation gives the numerical value of the number that was represented by the letter, and the solution of the problem may be completed.

Illustration

Jack and Harry together had \$19.64. Jack gave Harry 16 cents, and Jack then had exactly three times as much as Harry. How much did each have at first?

Let x = number of cents Harry had at first;
then $1964 - x$ = number of cents Jack had at first;
and $x + 16$ = number of cents Harry had after the transfer of 16 cents;
and $1964 - x - 16$ = number of cents Jack had after the transfer of 16 cents.

But after the transfer, Jack had three times as much as Harry.

Therefore $1964 - x - 16 = 3(x + 16)$.

The solution of the equation is $x = 475$, which means that Harry had \$4.75 and Jack \$14.89.

Proof

After the transfer of 16 cents from Jack to Harry, Harry had \$4.91, and Jack had $\$14.73 = 3 \times \$4.91$.

Exercise 27

1. Separate 100 into two parts, such that, if the larger part is divided by the smaller, the quotient is 3 and the remainder is 8.

2. A and B together have \$1.00, and A has 8 cents more than three times the amount that B has. How much does each have?

3. A had twice as much money as B. If B gave A 8 cents, A would have three times as much as B would then have left. How much did each have?

4. What number, x, must be subtracted from 100 so that the remainder shall be 8 more than the number x?

5. A number is just as much below 70 as it is above 48. Find the number.

6. The denominator of a fraction exceeds the numerator by 4. If the numerator is decreased by 4 and the denominator is increased by 1, the value of the fraction becomes $\frac{1}{2}$. Find the fraction.

7. The value of 17 coins consisting of dimes and quarters is $2.60. Find how many coins of each kind there are.

8. Find how many pounds of each of two grades of coffee should be used to make a mixture of 100 lbs. if the two grades are worth 15¢ and 27¢ a lb., and the mixture of 100 lbs. is worth $24.

9. $10,000 is invested, some at 4% and the remainder at 5%, so that the interest earned by the entire investment at the end of a year is $440. Find how much was invested at each rate of interest.

10. Dick is now twice as old as Mary, but $5\frac{1}{2}$ years ago he was three times as old as she was then. Find how old each is now.

11. Solve problems 1-10 without using algebra.

12. The sum of the squares of 3 successive integers exceeds the sum of the integers by 92. Find the integers.

13. The number x exceeds the number a by ax. Form the equation and solve it for x; for a.

14. An investment of p dollars at simple interest for n years at rate r amounts to s dollars. Express the statement as an equation and solve the equation for p; for n; for r.

15. A note for f dollars is payable at the end of n days with ordinary simple interest at rate r per annum and is discounted on the day it is drawn at the discount rate d. Show that the proceeds, p, may be found from $f\left(1 + \frac{rn}{360}\right)\left(1 - \frac{dn}{360}\right) = p$.

16. In example 15, show that, if $f = p$, then $\frac{1}{d} - \frac{1}{r} = \frac{n}{360}$.

17. The difference between the exact interest and the ordinary interest on p dollars for n days at rate r per annum is c dollars. Show that the relation between the letters is $npr = 5 \times 72 \times 73 \times c$.

18. A mixture consists of two grades of coffee worth a¢ and b¢ a lb., respectively. If there are n lbs. in the mixture and its value is nc¢, find how many lbs. of each grade it contains.

19. Six times the reciprocal of a number is added to the number and the sum is 5. Find the number.

20. In two fractions whose numerators are 1, one denominator exceeds the other by 3. The difference between the fractions is $\frac{1}{3}$ of the larger fraction. Find the two fractions.

21. A note having a maturity value of m dollars is discounted at rate d per annum n days before it is due. Show that the equivalent rate of interest, i, is found from $\frac{1}{i} = \frac{1}{d} - \frac{n}{360}$.

34. Simultaneous equations. The procedure followed in solving problems was to let one unknown number be represented by x and the other unknowns in terms of x. An equation was then formed in accordance with the conditions stated in the problem, and the equation was solved.

It is often more convenient to use 2, 3, or more letters to represent the various unknown numbers. But then as many equations must be formed as there are unknown letters used, and all the equations or conditions must be true at the same time or *simultaneously*. That is, values must be found for the letters which will reduce each equation to an identity.

The solution of a set of simultaneous equations is accomplished by combining them in such a way that one equation is found which contains only one letter, and we say that the other letters have been *eliminated*.

Illustrations

1. Solve

$$2x - 3y = 1 \tag{1}$$

$$5x + 2y = 31. \tag{2}$$

Since both members of any equation may be multiplied by any number without destroying the equality, multiply (1) by 2 and (2) by 3, and obtain

$$4x - 6y = 2 \tag{3}$$

$$15x + 6y = 93. \tag{4}$$

It is now possible to eliminate y by addition, since if equal numbers are added to the members of an equation, the equality is not destroyed.

The result of addition is

$$19x = 95. \qquad (5)$$

There is now a single equation containing only the unknown x, and $x = 5$.

Returning to the original equations (1) and (2), multiply (1) by 5 and (2) by 2 and obtain

$$10x - 15y = 5 \qquad (6)$$

$$10x + 4y = 62. \qquad (7)$$

It is now possible to eliminate x by subtraction, the result being

$$-19y = -57 \quad \text{and} \quad y = +3.$$

The values $x = 5$, $y = 3$ reduce the given equation to the identities $2 \times 5 - 3 \times 3 = 1$ and $5 \times 5 + 2 \times 3 = 31$.

Note: When we obtained $x = 5$, we had the right to replace x by 5 in (1), obtaining $10 - 3y = 1$, from which $y = 3$.

2. Solve

$$3x + 2y - 4z = 13 \qquad (1)$$

$$x - 3y + 2z = 7 \qquad (2)$$

$$2x + 4y - 3z = 13 \qquad (3)$$

The values of x, y, and z which will reduce these three equations to identities are found by eliminating two letters, as follows:

Eliminate z from (1) and (2), $5x - 4y = 27$ (4)

" z from (3) and (2), $7x - y = 47$ (5)

That is, the set of 3 equations with 3 unknowns has been replaced by a set of 2 equations with 2 unknowns.

Eliminate y from (4) and (5), $23x = 161$. (6)

The solution of (6) is $x = 7$, which reduces either (4) or (5) to $y = 2$. The values $x = 7$, $y = 2$ reduce any one of the given equations to $z = 3$. Hence the solution is $x = 7$, $y = 2$, $z = 3$.

The student may show that these values of x, y, and z make each of the given equations an identity.

3. A set of 4 simultaneous equations that contains the 4 unknowns x, y, z, w is solved in a manner similar to the method used for three unknowns.

Eliminate w from 2 equations and obtain an equation in x, y, z. Eliminate w again from 2 equations. Finally form a third equation by eliminating w from another pair. Care must be taken, however, that each of the given equations is used in the process of elimination.

From the three equations in x, y, z, eliminate z twice and obtain two equations in x, y, from which y may be eliminated and x found. Use this value of x in one of the equations in x, y and find y. Use the values of x and y in one of the equations in x, y, z and find z. Use the values of x, y, z in one of the equations in x, y, z, w and find w.

The method is perfectly general and is applicable to the solution of n simultaneous equations that contain n unknowns.

Exercise 28

Solve each of the following sets of simultaneous equations:

1. $x - y = 3$
   $x - 2y = 1$
2. $3x - 2y = 5$
   $x - 2y = 2$
3. $3x - 2y = 24$
   $2x - 3y = 11$
4. $2a - 3x = 7$
   $4a + x = 21$
5. $3a - b = 2$
   $2a + 5b = 7$
6. $a - 2y = 4$
   $3a + 2y = 16$
7. $4x + 7y = 41$
   $3x - 2y = 9$
8. $3x - 5y = 4$
   $2x + 3y = 9$
9. $3x + 5y = 8$
   $2x + 3y = 3$
10. $3x - 5y = -19$
    $2x + 3y = -19$
11. $x + y + z = 11$
    $x + y - z = 9$
    $x - y - z = 5$
12. $2x - 3y + z = 1$
    $3x - y + 2z = 13$
    $x + 2y - 3z = 4$
13. $2a + 3b + 4c = 28$
    $3a - 2b - c = 8$
    $a + b - 3c = -2$
14. $2x + 3y = 18$
    $3x + 2z = 24$
    $4y - 3z = -1$
15. $3b - 5a = 4$
    $4a - 2c = -2$
    $5c + 4b = 77$
16. $3r + s - 4t = 7$
    $4r + 2s - 6t = 10$
    $5r - 3s + 2t = 9$

17. If $y = ax + b$ and $y = 10$ when $x = 4$, and $y = -1$ when $x = \frac{1}{3}$, find the values of a and b.

18. If $y = ax^2 + bx + c$ and $y = 1$ when $x = 2$, $y = 8$ when $x = 3$, and $y = 13$ when $x = -2$, find a, b, c.

19. If $y = ax^3 + bx^2 + cx + d$ and $y = 3$ when x is 1 or -2, and $y = 7$ when $x = 2$ or -1, find a, b, c, d.

20. Solve
$$x - y + z - t = 4$$
$$2x + y - z - t = 3$$

$$3x - y + 2z + t = 16$$
$$x + 2y - z - t = 2.$$

35. Conditions for simultaneity. Either of the two equations $x + y = 5$ and $2x + 2y = 10$ may be obtained from the other. The two equations are said to be dependent one upon the other, and they really represent only one relation between x and y. A condition for simultaneity therefore is that the equations must be *independent.*

The two equations $x + y = 5$ and $x + y = 6$ contradict each other or are inconsistent. Another condition for simultaneity, therefore, is that the equations must be *consistent.*

When there are three equations with three unknowns, it is not easy to tell by inspection whether the equations are independent and consistent. In that case we proceed with the solution on the assumption that the equations are simultaneous. Then if the final result of elimination is an identity such as $0 = 0$, the equations are not independent; if elimination produces a result such as $0 = 3$, the equations are not consistent. In either case the equations are not simultaneous. If the equations are dependent, there may be any number of solutions; if the equations are inconsistent, there is no solution.

Exercise 29

Try to solve the following sets of equations:

1. $2x + 3y = 8$
$6x + 9y = 20$

2. $x + y + z = 8$
$3x - 2y - 2z = 9$
$4x - y - z = 17$

3. $x + y + z = 9$
$2x - y - 2z = 5$
$-x + 5y + 7z = 17$

4. $x + y + z = 8$
$3x - 2y - 2z = 9$
$7x - 3y - 3z = 26$

5. $14x - 49y = 35$
$10x - 35y = 35$

6. Show that in examples 2, 3, and 4, one of the equations can be obtained from the remaining two equations.

36. Problems. In the statement of a problem there may be several different unknown numbers and it may be easier to represent each of the unknowns by a different letter than to represent them all in terms of the same letter. For a solution to be possible, we must be able to form as many equations as there are unknowns, and the equations must be independent and consistent. If the given relations in the statement of the problem enable us to form a greater number of equations than there are unknown letters, some of the given relations are superfluous or they are inconsistent with the other given relations.

Illustrations

1. If the numerator and the denominator of a fraction are each increased by 1, the value of the fraction is $\frac{3}{4}$. If the numerator and the denominator are each decreased by 1, the value of the fraction is $\frac{2}{3}$. Find the fraction.

Solution

Let x = numerator, y = denominator, and $\frac{x}{y}$ = the fraction.

From the first statement,

$$\frac{x+1}{y+1} = \frac{3}{4}.$$

From the second statement,

$$\frac{x-1}{y-1} = \frac{2}{3}.$$

The solution of these equations is $x = 5$, $y = 7$. Hence the required fraction was $\frac{5}{7}$.

2. A number consists of 3 digits. If the digits are shifted, the digit in unit's place to 10's, the digit in 10's place to 100's place, and the digit in the 100's place to unit's place, the number is increased by 216. If the original number is divided by the sum of its digits, the quotient is

23 and the remainder is 12. The sum of the end digits is twice the middle one. Find the number.

Solution

Let x = the digit in 100's place, y = the digit in 10's place, z = the digit in unit's place.

Then the sum of the digits is $x + y + z$; the number is indicated by $100x + 10y + z$; and the result of shifting the digits is $100y + 10z + x$.

From the first statement,

$$100y + 10z + x = 100x + 10y + z + 216.$$

From the second statement,

$$100x + 10y + z = 23(x + y + z) + 12.$$

From the third statement,

$$x + z = 2y.$$

The solution of the three equations is $x = 3$, $y = 5$, $z = 7$, and the number is 357.

Exercise 30

1. A number consists of two digits. If the number is divided by the sum of its digits, the quotient is 6 and the remainder is 3. If the digits are interchanged, the new number is less than the original number by 18. Find the number.

2. The number of years in John's age is as much as the sum of Mary's and Bill's ages. Five years hence, twice the number of years in Mary's age will be 20 more than the sum of the ages of John and Bill at that time. Thirty years hence, twice Bill's age will be 5 years more than John's is then. How old is each now?

3. Henry, Frank, and Alice have together \$5.00. If Frank gives 30 cents to Henry and 45 cents to Alice, Frank will have 3 cents more than Henry and 14 cents more than twice as much as Alice. How much did each have at first?

4. A number consists of 3 digits. The number is 3 more than 31 times the sum of the digits. If the digits in unit's and 100's places are interchanged, the new number will exceed the original number by 297. The sum of the end digits exceeds 3 times the middle digit by 2. Find the number.

5. The unit's digit of a two-digit number exceeds the ten's digit by 5. If the digits are interchanged and the new number is divided by the sum of its digits, the quotient is 7 and the remainder is 3. Find the original number.

6. If the digits of a two-digit number are interchanged, the new number is less than the original one by 15 times the original unit's digit. The sum of the original number and the new number is 121. Find the original number.

7. The amount of water necessary to fill 3 jars of different sizes will fill the smallest jar 4 times; the largest jar twice, with 4 gallons to spare; and the second jar 3 times, with 2 gallons to spare. Find the capacity of each jar.

37. Roots. The values of x for different forms of equations are written as follows:

(a) $x^2 = 49$ gives $x = \sqrt{49}$, read "the square root of 49," and $x = 7$, since $7^2 = 49$. Another value of x is -7, since $(-7)^2 = 49$. Both results are indicated by writing $x = \pm\sqrt{49} = \pm 7$, read "x equals $+$ or $-$ 7."

$x^2 = 5$ gives $x = \pm\sqrt{5}$. The numerical value of $\sqrt{5}$ is between 2 and 3, since $2^2 = 4$, $3^2 = 9$, and 5 is between 4 and 9.

$x^2 = -3$ gives $x = \pm\sqrt{-3}$. Since the square of a $+$ or of a $-$ number gives a $+$ result, $\sqrt{-3}$ is neither a $+$ nor a $-$ number. The square root of a minus number is called an *imaginary number*, not because it does not exist, but because it cannot be represented by a point on the scale of real numbers. The algebraic sum of a real and an imaginary number, as $2 + \sqrt{-3}$, is called a *complex number*.

(b) $x^3 = 125$ gives $x = \sqrt[3]{125}$, read "the cube root of 125," and $x = 5$, since $5^3 = 125$. There are three different cube roots of 125, the remaining two being complex numbers. 5 is called the principal cube root of 125.

$x^3 = 50$ gives $x = \sqrt[3]{50}$. Since $3^3 = 27$ and $4^3 = 64$, and 50 is between 27 and 64, the cube root of 50 is between 3 and 4. We write $4 > x > 3$.

(c) $x^4 = 30$ gives $x = \sqrt[4]{30}$, the fourth root of 30, and x lies between 2 and 3.

38. Square root and cube root. Rules are easily dis-

covered for calculating the square root or the cube root of a number to any number of decimals.

(a) *Square root.* From the relation $\sqrt{a^2 + 2ab + b^2} = \sqrt{(a + b)^2} = a + b$, we formulate the rules for finding the square root.

The first term, a, of the answer is $\sqrt{a^2}$. Upon testing this result by squaring it, we find that there is still left $2ab + b^2$, from which the remaining term of the answer, b, must be found, and there should then be no remainder. Since $2ab + b^2 = b\,(2a + b)$, the term b is found by dividing $2ab + b^2$ by $2a + b$. The first part of the divisor, $2a$, is twice the answer found so far and is used as a trial divisor. That is, $2ab \div 2a = b$. The new term b is the second term of the answer and is also to be added to the trial divisor $2a$ in order to make the complete divisor $2a + b$.

Illustration

$$
\begin{array}{ll}
\sqrt{4x^4} = 2x^2 & \quad 2x^2 - 3x + 5 \\
 & \sqrt{4x^4 - 12x^3 + 29x^2 - 30x + 25} \\
(2x^2)^2 & \underline{4x^4} \\
\text{Remainder} & \quad\;\; -12x^3 + 29x^2 - 30x + 25 \\
\text{The trial divisor is } 4x^2 \text{ and} & \\
-12x^3 \div 4x^2 = -3x. & \\
\text{The complete divisor is } 4x^2 - 3x. & \\
(4x^2 - 3x)(-3x) & \quad\;\; \underline{-12x^3 + 9x^2} \\
\text{Remainder,} & \qquad\qquad + 20x^2 - 30x + 25 \\
\text{The trial divisor is } 4x^2 - 6x \text{ and} & \\
+20x^2 \div 4x^2 = 5. & \\
\text{The complete divisor is } (4x^2 - 6x + 5). & \\
(4x^2 - 6x + 5)(5) & \qquad\qquad \underline{+ 20x^2 - 30x + 25} \\
\text{Remainder} & \qquad\qquad\qquad\quad 0
\end{array}
$$

Hence $\quad \sqrt{4x^4 - 12x^3 + 29x^2 - 30x + 25} = 2x^2 - 3x + 5.$

Let us apply this method to find $\sqrt{120{,}936}$.

Since $300^2 = 90{,}000$ and $400^2 = 160{,}000$, 120,936 is written $300^2 + 30{,}936$. The setup is

		$300 + 40 + 7 + .7$
		$\sqrt{120936}$
300^2		90000
		30936
Trial divisor	600	
$30{,}936 \div (600+) =$	40	
Complete divisor	640	
640×40		25600
Remainder		5336
Trial divisor	680	
$5336 \div (680+) =$	7	
Complete divisor	687	
687×7		4809
Remainder		527.00
Trial divisor	694	
$527 \div (694+) =$	.7	
Complete divisor	694.7	
$694.7 \times .7$		486.29
Remainder		40.71

The process may be continued as far as we please.

The setup in abbreviated form appears thus:

$$\begin{array}{r|l}
 & 3\quad 4\quad 7.\ \ 7\quad 5 \\
 & \sqrt{12\ 09\ 36.00\ 00} \\
 & 9 \\
6\cancel{4} & 3\ 09 \\
 & 2\ 56 \\
68\cancel{7} & 53\ 36 \\
 & 48\ 09 \\
694\cancel{7} & 5\ 27\ 00 \\
 & 4\ 86\ 29 \\
6954\cancel{5} & 40\ 71\ 00 \\
 & 34\ 77\ 25 \\
 & 5\ 93\ 75
\end{array}$$

Hence $\sqrt{120{,}936} = 347.75\ldots$

The given number is separated into sets of two figures each, the decimal point being one of the separation points. The divisor consists of two parts, the trial divisor and the new term of the answer, the crossed-out figure. For each set of figures of the number there is one figure in the square root.

(b) *Cube root.* Since $\sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3} = \sqrt[3]{(a + b)^3} = a + b$, the first term of the answer is $\sqrt[3]{a^3} = a$, which when cubed and subtracted leaves $3a^2b + 3ab^2 + b^3 = b(3a^2 + 3ab + b^2)$. In order to find b, the second term of the answer, the remainder must be divided by $3a^2 + 3ab + b^2$. The trial divisor is $3a^2$, three times the square of the answer. Then $3a^2b \div 3a^2$ gives b, the second term of the answer. The divisor is now completed by adding to $3a^2$ the two terms $3ab$ (three times the old answer times the new term) and b^2 (the square of the new term), and there is no remainder.

Applying these rules to find $\sqrt[3]{15}$, we note that $15 = 2^3 + 7$, and the first part of the answer is $\sqrt[3]{2^3}$ and the remainder is 7. The trial divisor is $3 \times 2^2 = 12$, and $7 \div 12^+$ is .4 since .5 is found to be too large. The complete divisor is $3 \times 2^2 + 3 \times 2 \times .4 + (.4)^2 = 14.56$, and $14.56 \times .4 = 5.824$, leaving a remainder of 1.176, and so on. The setup is

		2 . 4 6 6
		$\sqrt[3]{15 \;.000\;000\;000}$
		8
Trial divisor 3×2^2 =	12	7 .000
$3 \times 2 \times .4$ =	2.4	
$(.4)^2$ =	.16	
Complete divisor	14.56	5 .824
		1 .176 000
Trial divisor $3 \times (2.4)^2$ =	17.28	
$3 \times 2.4 \times .06$ =	.432	
$(.06)^2$ =	.0036	
Complete divisor	17.7156	1 .062 936
		.113 064 000
Trial divisor 3×2.46^2 =	18.1548	
$3 \times 2.46 \times .006$ =	.04428	
$(.006)^2$ =	.000036	
Complete divisor	18.199116	.109 194 696
		.003 869 304

Hence $\sqrt[3]{15} = 2.466 \ldots$

In cube root the given number is separated into sets of 3 figures, the decimal point being a separation point, and the divisor consists of three parts. For each set of three figures in the number there is one figure in the cube root.

The values of $\sqrt{n}$, $\sqrt{10n}$, $\sqrt[3]{n}$, $\sqrt[3]{10n}$, and $\sqrt[3]{100n}$ are shown in Table I. Thus on the line $n = 34$, we find $\sqrt{34} = 5.83095$, $\sqrt{340} = 18.4391$, $\sqrt[3]{34} = 3.23961$, $\sqrt[3]{340} = 6.97953$, $\sqrt[3]{3400} = 15.0369$.

To find the value of $\sqrt{.034}$, note that by the method of square root the number is written .03 40 00 00 00 00, that there are 6 figures in the answer, that the decimal point is placed before the first figure of the answer, and that the sequence of figures in the answer is the same as for $\sqrt{340}$. Hence $\sqrt{.034} = .184391$.

To find the value of $\sqrt[3]{3.4}$, the number is written 3. 400 000 000 000 000. The sequence of figures in the answer is the same as for $\sqrt[3]{3400}$, but the decimal point is placed after the first figure. Hence $\sqrt[3]{3.4} = 1.50369$.

The table therefore enables us to find the square root or the cube root of numbers such as 3400, 34, 3.4, .34, .034.

We shall see (page 90) how to find the square root or the cube root of a number such as 3.46.

Exercise 31

1. Find the square root of each of the following:

(a) $4x^4 - 12x^3 + 25x^2 - 24x + 16$
(b) $x^4 - 2x^3 + 3x^2 - 2x + 1$
(c) $x^6 - 8x^5 + 12x^4 + 22x^3 - 20x^2 - 12x + 9$

2. Square each of the following numbers and find the square root of the resulting number: (a) 137; (b) 13.7; (c) 1.37.

3. Find the value of each of the following to 4 figures and check with Table I:

(a) $\sqrt{2}$	(d) $\sqrt{6}$	(g) $\sqrt{20}$
(b) $\sqrt{3}$	(e) $\sqrt{8}$	(h) $\sqrt{125}$
(c) $\sqrt{5}$	(f) $\sqrt{10}$	(i) $\sqrt{.2}$

4. Verify the following by using the results of example 3.

(a) $\sqrt{6} = \sqrt{3} \times \sqrt{2}$; (b) $\sqrt{8} = \sqrt{4} \times \sqrt{2}$; (c) $\sqrt{10} = \sqrt{5} \times \sqrt{2}$; (d) $\sqrt{125} = \sqrt{25} \times \sqrt{5}$; (e) $\sqrt{20} = 2\sqrt{5}$; (f) $\sqrt{.2} = .1\sqrt{20}$.

5. Find the values of the following by means of Table I:

(a) $\sqrt{4.8}$; (b) $\sqrt[3]{4.8}$; (c) $\sqrt{.0048}$; (d) $\sqrt[3]{.048}$; (e) $\sqrt{48000}$; (f) $\sqrt[3]{480000}$.

39. Degree of an algebraic equation. An algebraic polynomial P in x consists of a number of terms containing positive integral powers of x with known coefficients. The polynomial is said to be of degree 1, 2, 3, 4, and so forth, if the largest exponent of x is 1, 2, 3, 4, and so forth. Thus $P = x^3 - \frac{2}{3}x^2 - 4x + 7$ is an algebraic polynomial of the third degree. If $P = 0$, we have an equation of the same degree as P. The solution of an equation is more difficult as the degree of the equation is higher. It is often possible however to factor P by the factor theorem into first degree factors, and an equation of the 2nd, 3rd, or 4th degree may have simple solutions.

40. Quadratic equations. An equation of the 2nd degree in x, such as $3x^2 - 5x - 3 = 0$, is called a *quadratic equation.* To solve the quadratic equation $ax^2 + bx + c = 0$ means that we must express x in terms of a, b, and c. We can do this by writing the following successive equivalent forms:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ ax^2 + bx &= -c \\ 4a^2x^2 + 4abx &= -4ac \\ 4a^2x^2 + 4abx + b^2 &= b^2 - 4ac \\ (2ax + b)^2 &= (b^2 - 4ac). \end{aligned}$$

The last form enables us to write:

$$\begin{aligned} 2ax + b &= \pm\sqrt{(b^2 - 4ac)} \\ 2ax &= -b \pm \sqrt{(b^2 - 4ac)} \\ x &= \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}, \end{aligned}$$

which is the required solution.

Illustration

To solve $3x^2 - 5x - 3 = 0$, we may repeat the steps taken in the general case and obtain the successive forms:

$$\begin{aligned} 3x^2 - 5x - 3 &= 0 \\ 3x^2 - 5x &= 3 \\ 36x^2 - 60x &= 36 \\ 36x^2 - 60x + 25 &= 36 + 25 = 61 \\ (6x - 5)^2 &= 61 \\ 6x - 5 &= \pm\sqrt{61} \\ 6x &= 5 \pm \sqrt{61} \\ x &= \frac{5 \pm \sqrt{61}}{6}; \end{aligned}$$

or, we may compare $ax^2 + bx + c = 0$

with $3x^2 - 5x - 3 = 0$

and note that $a = 3$, $b = -5$, $c = -3$.

Since $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-3)}}{2(3)} = \frac{5 \pm \sqrt{61}}{6}.$$

$$\sqrt{61} = 7.810 \text{ and } x = \frac{5 \pm 7.810}{6}.$$

The two values of x, or the roots of the equation, are

$$x = \frac{5 + 7.810}{6} = 2.135, \text{ and } x = \frac{5 - 7.810}{6} = -.468.$$

Exercise 32

Solve each of the following quadratic equations exactly or to 3 decimals.

1. $2x^2 - 5x - 3 = 0$
2. $3x^2 + 7x = 2$
3. $5x^2 - 1 = x$
4. $x^2 = x + 1$
5. $3x + 3 = 2x^2$
6. $2x^2 + 3 = 5x$
7. $x + 1 = \frac{1}{x}$
8. $\frac{x^2}{3} - x = 1$
9. $9x - \frac{5}{x} = 2$
10. $\frac{5x}{x - 2} = x + 2$

Solve each of the following equations with the aid of the factor theorem.

11. $x^3 - 2x^2 - 2x + 3 = 0$
12. $x^4 - 6x^3 + 7x^2 + 14x - 24 = 0$

41. The quadratic equation and its roots. Many noteworthy relations exist between the quadratic equation $ax^2 + bx + c = 0$ and its roots $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

(1) If $b^2 - 4ac = 0$, the roots are real and equal, each root being $\dfrac{-b}{2a}$.

(2) If $b^2 - 4ac > 0$, the roots are real numbers.

(3) If $b^2 - 4ac < 0$, the roots are complex numbers.

(4) The sum of the roots is $\dfrac{-b}{a}$.

(5) The product of the roots is $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence if the equation is written $x^2 + \dfrac{bx}{a} + \dfrac{c}{a} = 0$, the coefficient of x with the sign changed is the sum of the roots, and the constant term is the product of the roots.

Thus in the equation $3x^2 - 7x - 30 = 0$, the sum of the roots is $\dfrac{+7}{3}$ and the product is $\dfrac{-30}{3}$.

(6) If the roots are given, say -3 and $+5$, the quadratic equation may be written, for the sum of the roots is $+2$ and the product is -15. Hence the equation is $x^2 - 2x - 15 = 0$.

Illustrations

1. One root of $kx^2 - 4x - 3 = 0$ exceeds the other root by 2. Find k and the roots.

Let r = one root, and $r + 2$ = the other root.

Then $r + (r + 2) = +\dfrac{4}{k}$ and $r(r + 2) = -\dfrac{3}{k}$.

From the equations $2r + 2 = \frac{4}{k}$ and $r^2 + 2r = -\frac{3}{k}$, we may eliminate k by dividing one equation by the other, obtaining $\frac{2r + 2}{r^2 + 2r} = -\frac{4}{3}$, which reduces to $2r^2 + 7r + 3 = 0$. Hence $r = -\frac{1}{2}$ or -3. If $r = -\frac{1}{2}$, $r + 2 = \frac{3}{2}$, and $k = 4$. If $r = -3$, $r + 2 = -1$, and $k = -1$.

Hence k may have two values—namely, 4 or -1. For $k = 4$, the roots are $-\frac{1}{2}$ and $\frac{3}{2}$, one exceeding the other by 2; for $k = -1$, the roots are -3 and -1, one exceeding the other by 2.

2. Solve $2x^2 - 16x + 19 = 0$ and check the roots. The roots are

$$x = \frac{16 \pm \sqrt{256 - 152}}{4} = \frac{16 \pm \sqrt{104}}{4} = \frac{16 \pm 10.198}{4}$$

$= \frac{26.198}{4}$ or $\frac{5.802}{4}$, and $x = 6.5495$ or 1.4505.

Check: The sum of the roots should be $+\frac{16}{2} = 8$, and it is 8. The product of the roots should be $+\frac{19}{2} = 9.5$, and it is 9.5001.

Exercise 33

1. Without calculating the roots, state the sum of the roots and the product of the roots for each of the following equations:

(a) $x^2 - 4x + 4 = 0$
(b) $x^2 + x + 1 = 0$
(c) $3x^2 - 5x - 2 = 0$
(d) $x^2 - x + 1 = 0$
(e) $3x^2 + 5x - 2 = 0$
(f) $5x^2 - 7x + 4 = 0$

2. Write the quadratic equation whose roots are:

(a) 2, 3
(b) -1, 4
(c) $\frac{1}{2}$, $\frac{1}{3}$
(d) 3, $\frac{1}{3}$
(e) -1, -2
(f) -3, $+3$

3. State the kind of roots that result from each of the following equations:

(a) $x^2 - 4x + 4 = 0$
(b) $3x^2 - 5x - 2 = 0$
(c) $3x^2 + 5x + 2 = 0$
(d) $kx^2 - 2kx + 3 = 0$
(e) $x^2 + x + 1 = 0$
(f) $x^2 - x + 1 = 0$
(g) $5x^2 - 7x + 4 = 0$
(h) $x^2 + 3x - c = 0$

4. In example 3(d), find the value of k so that: (a) the roots shall be real; (b) the roots shall differ by 4.

5. In example 3(h), find the value of c so that: (a) the roots shall be

real; (b) the roots shall differ by 5; (c) one root shall be twice the other root.

6. Find k from $kx^2 - 8x + 3 = 0$ so that: (a) the roots shall be real; (b) one root shall be three times the other root; (c) the roots shall differ by 1.

7. Find a number which exceeds its reciprocal by $\frac{5}{6}$.

8. The sum of a number and its reciprocal is $2\frac{1}{6}$. Find the number.

9. An article is marked \$60 and the net price after the discounts is \$28.80. Two successive discounts are allowed, one being twice the other. Find the separate discounts.

10. An article is marked \$60, and the net price after discounts is \$21. Two successive discounts are allowed, the sum of the discounts being 80%. Find the separate discounts.

11. A number consists of two digits whose sum is 4. The square of the number exceeds 50 times the unit's digit by 19. Find the number.

12. An army 1 mile long is marching at a steady rate of 4 miles per hour. A mounted soldier at the rear end of the line rides to the front and then returns to his place. He finds that he is now at the point where the head of the line was when he left his place. If no time was lost in making stops and he rode at a uniform speed, what was his speed?

13. One root of $kx^2 - 16x + 15 = 0$ exceeds the other root by 1. Find k and the roots.

14. The roots of $x^2 - x - 1 = 0$ being represented by h and k, $h + k = +1$ and $hk = -1$. Show without finding h and k that:

(a) $\frac{1}{h} + \frac{1}{k} = -1$; (b) $h^2 + k^2 = 3$; (c) $\frac{1}{h^2} + \frac{1}{k^2} = 3$; (d) $h^3 + k^3 = 4$;

(e) $\frac{h}{k} + \frac{k}{h} = -3$; (f) $h^4 + k^4 = 7$.

CHAPTER IV

GEOMETRIC RELATIONS

42. How a graph is made. A manufacturer finds that each of a number of machines can produce from 10 to 100 articles a day. The entire product is sold at \$2 an article. The cost, the sales, and the profit are:

Articles	*Cost*	*Sales*	*Profit*
10	\$ 30	\$ 20	\$–10
20	45	40	– 5
30	58	60	2
40	70	80	10
50	81	100	19
60	95	120	25
70	112	140	28
80	131	160	29
90	153	180	27
100	176	200	24

These results are visualized by means of a drawing. Two lines, called *axes*, are drawn at right angles to each other, as in Fig. 3. Beginning at O, the crossing place of the axes, equal spaces are marked off on both axes, and the division points are numbered consecutively. The numbers on the horizontal axis indicate the number of articles, and those on the vertical axis, dollars.

The point A is the graphic representation of the statement that 10 articles cost \$30, and each of the points B, C, D, and so forth is the graphic representation of a similar statement. The points A, B, C, D, and so on are joined either by straight lines or by a smooth curve, so that the eye may easily follow the successive statements, and the resulting figure is called the *cost graph*. The sales graph and the profit graph are drawn in a similar manner.

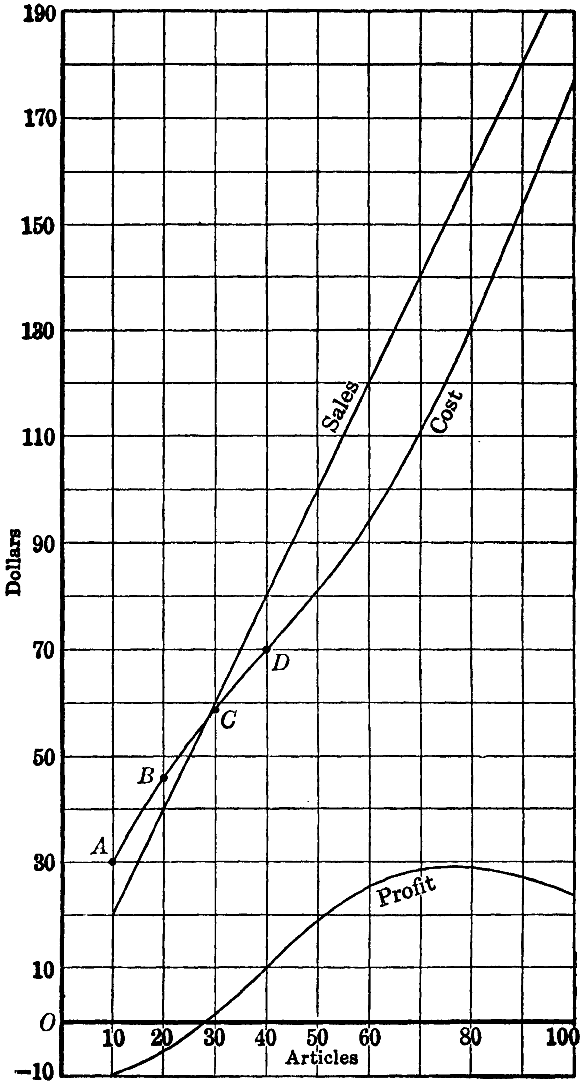

190
170
150
130
110
90
70
50
30
10
O
-10
Dollars
Sales
Cost
Profit
A
B
C
D
10
20
30
40
50
60
80
100
Articles

Fig. 3.

Exercise 34

1. What statement does each point of the graph represent?

2. For how many articles is the cost the same as the selling price? How much is the cost?

3. For how many articles is the cost $100?

4. For how many articles is the profit $20?

5. What is the most profitable output of a machine?

6. Can the profit graph be drawn from the cost and the sales graphs without referring to the table of profits?

7. Can the cost graph be drawn from the sales and the profit graphs without referring to the table of costs?

43. Graph of an equation. An equation in which y is expressed in terms of x, such as $y = 3x^2 - 7x + 3$, is represented graphically as follows: Assign successive values to x and calculate y in each case. The result is a table of corresponding values of x and y which may be extended as far as we please.

x	-2	-1	0	1	2	3	4
y	29	13	3	-1	1	9	23
Point	A	B	C	D	E	F	G

We now call the horizontal axis the "x axis" and the vertical axis the "y axis." Values of x are measured horizontally, toward the right if they are plus and toward the left if they are minus. Values of y are measured vertically, upward if they are plus and downward if they are minus.

Two corresponding values of x and y enable us to locate a point. The position of the point B is indicated by $(-1, +13)$. The first number, -1, the value of x, is called the "abscissa of B," and the second number, $+13$, the value of y, is called the "ordinate of B." The abscissa and the ordinate of a point are called the *coördinates* of the point, the axes are called the *coördinate axes*, and the point

O where the coördinate axes cross is called the *origin of coördinates.* The point B is located 1 unit to the left of the origin in the direction of the x axis and 13 units upward in the direction of the y axis. The other points of the table are plotted similarly.

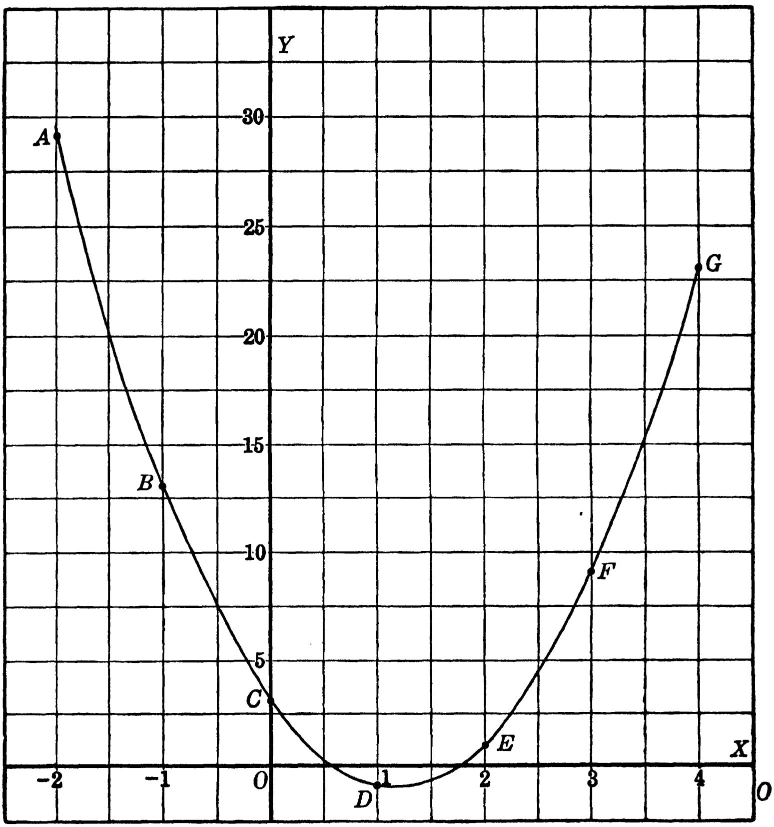


Fig. 4.

Additional points may be found between D and E by setting $x = 1.1$, 1.2, 1.3, and so forth. Similarly, addi-

tional points may be found between A and B, B and C, and so on. The result is a set of points that plainly indicate a curve, and the smooth curve drawn through the successive points is the graph of the equation.

Note that every point on the curve in Fig. 4, the graph of the equation $y = 3x^2 - 7x + 3$, has a real meaning. That is, there are really corresponding values of x and y obtainable from the equation to determine the point. Such is not the case in Fig. 3, page 69. A point on the profit graph, where $x = 63.5$, for example, has no real meaning.

Exercise 35

Answer the following questions by reference to Fig. 4, the graph of $y = 3x^2 - 7x + 3$:

1. What is the value of y if $x = 2.5$?

2. If x has any specified value, how many values does y have?

3. For what value of x is y least?

4. For what value of x is $y = 10$?

5. What is the value of x if $y = 0$?

6. What is the value of x if $y = -3$?

7. If y has any specified value, how many values does x have?

8. How would the values of y in the equation $y = 3x^2 - 7x$ differ from the values of y in the equation $y = 3x^2 - 7x + 3$?

9. How should the graphs of the two equations in example 8 differ?

10. How may the equation $x = 3y^2 - 7y + 3$ be obtained from the equation $y = 3x^2 - 7x + 3$?

11. How should the graphs of the two equations in example 10 differ?

44. Geometric relations. Certain geometric relations must necessarily be introduced into a discussion of graphs. In the following discussions, all the lines or curves in any

figure are assumed to be on a flat surface, such as the surface of the paper, called a *plane*, and are called *plane figures*.

(1) *Angles.* Fig. 5 indicates that a line AD is rotated about a point A which remains fixed. Every point of the rotating line describes the circumference of a circle whose center is A. At any position of the rotating line, AR, a part of a circumference, BP, CQ, or DR, called an *arc*, has been described. Each of these arcs is the same fractional part of the entire circumference on which it is found. If DR is $\frac{1}{6}$ of its circumference, CQ is also $\frac{1}{6}$ of its circumference, and BP is $\frac{1}{6}$ of its circumference.

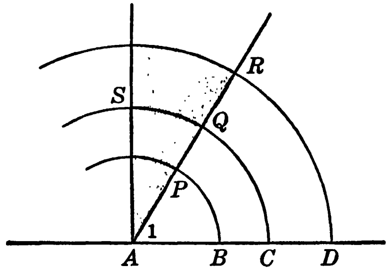


Fig. 5.

A circumference is divided into 360 equal parts called *degreees*, and each of the arcs BP, CQ, DR contains the same number of degrees. If smaller units are needed, each degree is divided into 60 minutes and each minute into 60 seconds.

The lines AD and AR are drawn in different directions from A, and the *difference* in direction is called the *angle* between AD and AR. The angle is written "angle DAR" or "angle 1." The number of degrees in angle 1 is precisely the same as the number of degrees in any one of the arcs BP, CQ, DR. It is written in such a form as "angle 1 = $50° 20'30''$" (read "50 degrees, 20 minutes, and 30 seconds").

If the arc CS is one fourth of its circumference, it contains 90°. The two lines AC and AS make the angle CAS, which is 90°, or a *right angle*, and the lines AC and AS are said to be perpendicular to each other. An angle less than 90° is an *acute angle;* an angle greater than 90° is an *obtuse angle.*

(2) *Triangles.* A closed figure bounded by three straight lines, Fig. 6, is called a *triangle.* Every triangle has three angles, whose sum is 180°. If two triangles have two angles of one triangle equal respectively to two angles of the other triangle, the third angles are equal.

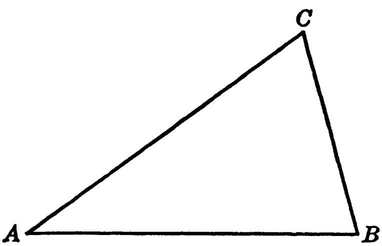


Fig. 6.

If one of the angles is a right angle, 90°, the triangle is called a *right triangle* and the side opposite the right angle is called the *hypotenuse.* In a right triangle the sum of the acute angles must be 90°. If two right triangles have an acute angle of one triangle equal to an acute angle of the other triangle, the third angles are equal.

Two triangles are called *congruent* if one can be made to coincide with the other. Two triangles are congruent if they have: (a) two sides and the included angle of one triangle equal respectively to two sides and the included angle of the other; (b) two angles and the included side of one triangle equal respectively to two angles and the included side of the other; (c) the three sides of one triangle equal respectively to the three sides of the other.

A triangle in which two sides are equal in length is called *isosceles.* In an isosceles triangle the angles opposite the equal sides are equal and the line that bisects the third angle divides the isosceles triangle into two congruent right triangles.

A triangle in which the three sides are equal in length is called *equilateral.* Since an equilateral triangle is also isosceles, the three angles are equal and each angle is 60°.

45. Similar triangles. Two triangles in which the three angles of one are equal, respectively, to the three angles of

the other are called *similar triangles.* In Fig. 7, equal angles are marked with the same number, and the triangles

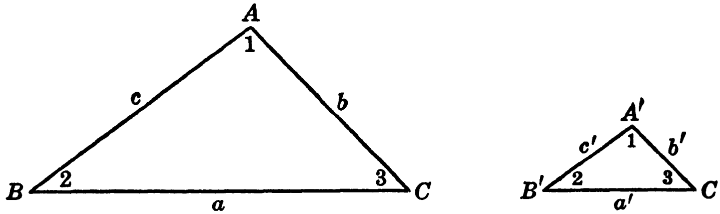


Fig. 7.

are similar. The sides are marked so that side a is opposite angle A, and so forth.

One of the triangles is a small picture of the other, and the sides that lie in corresponding positions, opposite equal angles, are proportional. That is,

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

If, in the similar triangles of Fig. 7, $a = 8$ inches and $a' = 2$ inches, not only is a four times as long as a', but also $b = 4b'$ and $c = 4c'$. The two triangles may be pictures of the same object drawn to different scales.

46. Right triangle relations. In the right triangle ABC, Fig. 8, the line CD is drawn at right angles to AB. Two new triangles are formed, ACD and BCD, which are similar to each other and also similar to the original triangle, ABC.

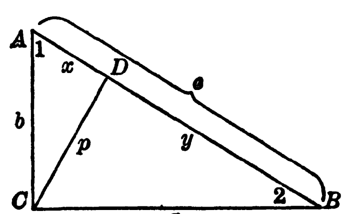


Fig. 8.

The reason for the similarity becomes evident if the triangles are drawn separately as in Fig. 9. The second triangle and the first have two angles of one equal to two angles of the other. The third

angles must therefore be equal, and the third angle of ACD should be marked with the number 2.

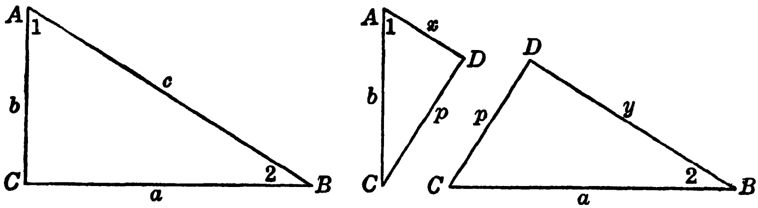


Fig. 9.

The third triangle and the first have two angles of one equal to two angles of the other. The third angles must therefore be equal, and the third angle of BCD should be marked with the number 1.

The second and third triangles are now seen to have the three angles of one equal to those of the other.

The following very important relations are now found:

$$\frac{x}{p} = \frac{p}{y}, \text{ or } p^2 = xy;$$

$$\frac{c}{b} = \frac{b}{x}, \text{ or } b^2 = cx;$$

$$\frac{c}{a} = \frac{a}{y}, \text{ or } a^2 = cy.$$

Adding the last two equations gives the well-known relation

$$a^2 + b^2 = cy + cx = c(y + x) = c \cdot c = c^2.$$

That is, the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. Hence if the lengths of the sides of a triangle are given, the angle opposite any side is 90°, greater than 90°, or less than 90°, according as its square is equal to, greater than, or less than the sum of the squares of the other two sides. Thus if the sides of a triangle are 4, 7, 9, the squares of the sides

are 16, 49, 81. Since $81 > 16 + 49$, the angle opposite the side whose length is 9 is obtuse.

Exercise 36

1. The angles of a triangle are designated by x, $2x$, and $3x$. How many degrees are there in each angle?

2. In a right triangle, one of the acute angles is three times as large as the other. How many degrees are there in each?

3. Two lines cross at an angle of 75°. What other angle do the lines make with each other?

4. The triangles in Fig. 10 are similar:

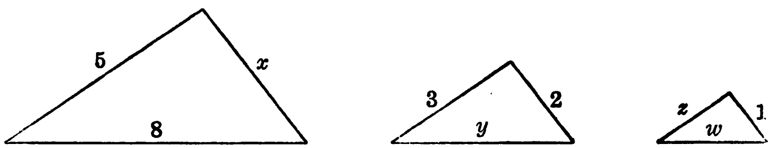


Fig. 10.

Calculate the lengths x, y, z, w.

5. In Fig. 11, AC is perpendicular to BE and to CD. If $AB = 3$ and $BC = 2$, what is the relation between BE and CD? Between AD and AE?

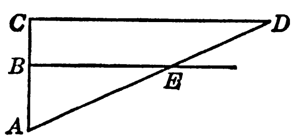


Fig. 11.

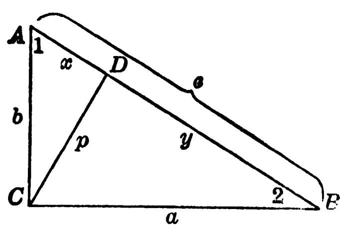


Fig. 8.
(Repeated from p. 75)

6. In the right triangle ABC, Fig. 8, CD is perpendicular to AB, $AC = 5$, $AD = 3$. Find the lengths of DB and CD.

7. In Fig. 12, a square, each side of which is 8 inches, is cut into four pieces as indicated. These four pieces are rearranged into what appears

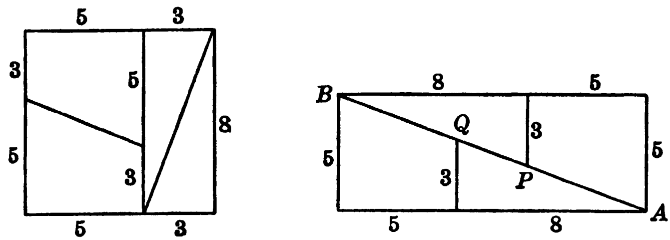


Fig. 12.

to be a rectangular figure 5 inches by 13 inches. Show that $APQB$ is not really a straight line.

8. In Fig. 13, all the angles are right angles and $OC = 2$, $OD = 3$, $CA = 5$, $DB = 12$. How far from O is the line OX crossed by each of the straight lines AD, CB, AB? Where does OA cross DB? Where does BO cross AC?

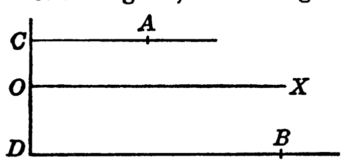


Fig. 13.

47. Area. A plane figure bounded by straight lines is called a *polygon*, and the surface included by the lines is

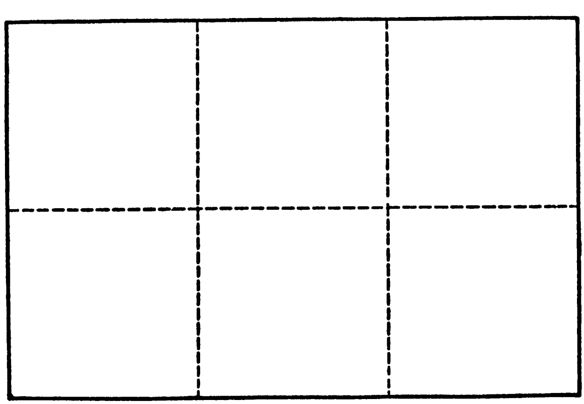


Fig. 14.

the area of the polygon. A unit of area is a *square*, each side of which is one unit long. A *rectangle* is a four-sided polygon whose angles are right angles. The rectangle in Fig. 14, 3 inches long and 2 inches wide, is divided by the dotted lines into 6 unit squares, and the area is 6 square inches. In general, if the dimensions of a rectangle are a and b, the area is ab.

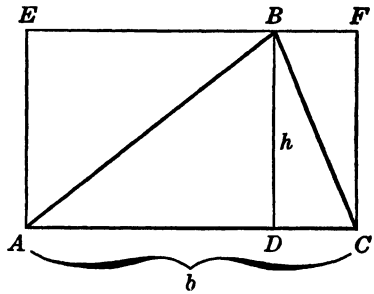


Fig. 15.

In the triangle ABC, Fig. 15, AC is called the *base*, and BD, the perpendicular to AC from B, is called the *altitude*. It is evident from the figure that the area of the rectangle $ACFE$ is twice the area of the triangle ABC. Hence the area of a triangle whose base is b and whose altitude is h is given by the formula

$$\text{Area} = \tfrac{1}{2}bh$$

Any polygon can be divided into triangles and its area can be calculated.

The area of a triangle whose sides are 8, 10, 12, is calculated as follows:

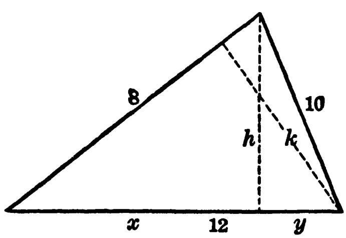


Fig. 16.

Let h be the altitude drawn to the longest side, 12, dividing

it into two parts, x and y, so that $x + y = 12$. In the right triangles,

$$h^2 = 8^2 - x^2 \text{ and } h^2 = 10^2 - y^2$$

Hence $8^2 - x^2 = 10^2 - y^2$ and $y^2 - x^2 = 10^2 - 8^2$

Factoring gives $(y + x)(y - x) = (10 + 8)(10 - 8)$,

or $\qquad 12(y - x) = 36$

and $\qquad y - x = 3.$

But $\qquad y + x = 12.$

Hence $y = 7\frac{1}{2}$, $x = 4\frac{1}{2}$, and $h^2 = 8^2 - x^2 = 12\frac{1}{2} \times 3\frac{1}{2} = \frac{25}{4} \times 7.$

Therefore, $h = \sqrt{\frac{25}{4} \times 7} = \frac{5}{2}\sqrt{7}.$

Area $= \frac{1}{2}$ base $\times$ altitude $= \frac{1}{2} \times 12 \times \frac{5}{2}\sqrt{7} = 15\sqrt{7}.$

Note that the altitude, k, to the side 8 may now be found from Area $= \frac{1}{2} \times 8 \times k = 15\sqrt{7}$. That is, $k = \frac{15}{4}\sqrt{7}.$

The general formula for the area of a triangle,

$$\text{Area} = \sqrt{s(s - a)(s - b)(s - c)},$$

where a, b, c are the lengths of the three sides and $a + b + c = 2s$, may be obtained by this method.

Exercise 37

1. Find the area of the equilateral triangle each of whose sides is 6 inches.

2. Find the area of the isosceles triangle each of whose equal sides is 8 inches and the remaining side is 4 inches.

3. Find the area of the triangles whose sides are (a) 6, 8, 10; (b) 5, 7, 8; (c) 6, 11, 13.

4. A *trapezoid* is a four-sided figure, two sides of which are parallel. Prove that the area of a trapezoid is found by multiplying half the sum of the parallel sides by the perpendicular distance between these sides.

48. Slope. A straight line extends indefinitely. Two lines on the plane of the paper either cross at some point and form an angle or never cross and are *parallel* to each

other. We say that two parallel lines go in the same direction, while two lines that cross go in different directions. The angle that two intersecting or crossing lines make is a measure of the difference in their directions. The direction of a line is usually compared with that of a horizontal line.

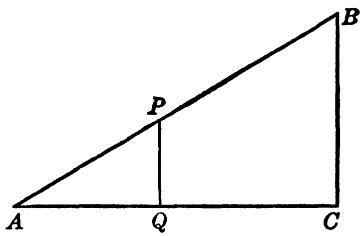


Fig. 17.

In Fig. 17, AC is a horizontal line, AB is an oblique line, and AB makes the angle CAB with AC. If BC is drawn perpendicular to AC, we say that to go from A to B we *rise* (move vertically upward) the distance CB and we *advance* (move horizontally toward the right) the distance AC. The fraction $\frac{CB}{AC}$ or $\frac{\text{rise}}{\text{advance}}$ is called the *slope* of AB.

The slope of AB may be determined by drawing the perpendicular PQ instead of BC. The rise then is QP, the advance is AQ, and the slope is $\frac{QP}{AQ}$.

But the triangles PQA and BCA are similar, and $\frac{CB}{CA} = \frac{QP}{AQ}$. That is, the slope of the line AB is the same whether we go from A to P or from A to B.

When points are referred to coördinate axes, say $A(3, 7)$, $B(5, 8)$, $C(10, 7)$, $D(5, 1)$, the first number in each case shows how far to the right of the y axis the point is, and the second number shows how far it is above the x axis. We can therefore determine the slope of a line that joins two points from the coördinates of these points without plotting them.

Proceeding from A to B, the rise is 1, the advance is 2, and the slope of AB is $\frac{1}{2}$.

Proceeding from B to C, we do not move upward but downward, and we say that the rise is -1. The advance

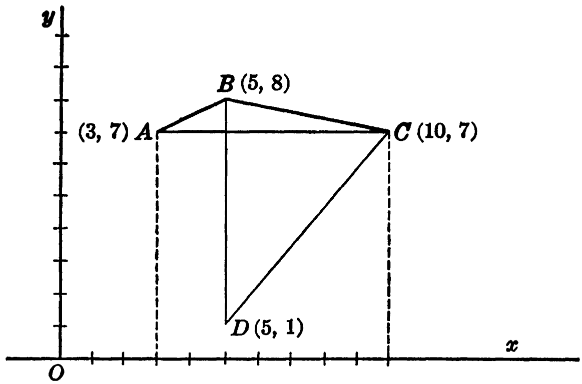


Fig. 18.

is 5, and the slope of BC is $-\frac{1}{5}$.

Proceeding from C to D, the rise is -6, the advance is -5, and the slope of CD is $\frac{-6}{-5}$, or $\frac{6}{5}$.

Proceeding from D to C, the rise is $+6$, the advance is $+5$, and the slope is $\frac{6}{5}$. That is, the slope of CD is the same whether we proceed from C to D or from D to C.

Proceeding from A to C, the rise is 0, the advance is 7, and the slope of AC is $\frac{0}{7}$, or 0. That is, the slope of a horizontal line is 0.

Proceeding from B to D, the rise is -7, the advance is 0, and the slope of BD is $-\frac{7}{0}$, indicated by ∞ (infinity). That is, BD is a vertical line, perpendicular to the horizontal axis.

In general, if two points P and Q are $P(x_1, y_1)$, $Q(x_2, y_2)$, the slope of the straight line PQ is $\frac{y_2 - y_1}{x_2 - x_1}$, or $\frac{y_1 - y_2}{x_1 - x_2}$.

Exercise 38

1. Given the points indicated:

Point	A	B	C	D	E	F
x	1	3	4	5	3	-2
y	2	-1	1	3	6	-4

(a) Find the slopes of AB, AC, AD, and so on—15 different possible lines.

(b) Show that the lines AB and DE are parallel.

(c) Show that B, C, D are collinear—that is, that they are on a straight line.

(d) Show that A, E, F are collinear.

(e) How is the line BE situated?

2. Given three points P, Q, R that lie on a straight line whose slope is $\frac{1}{2}$. If the rise from P to Q is $\frac{2}{3}$ of the rise from P to R, how does the advance from P to Q compare with the advance from P to R? How does the length of PQ compare with the length of PR?

3. Given the points $A(1,2)$ and $B(10,17)$.

(a) A point P is on the line AB, and the x value of P is 4. Find the y value of P.

(b) A point Q is on the line AB, and the y value of Q is 5. Find the x value of Q.

(c) A point R is on the line AB midway between A and B. Find the coördinates of R.

4. The points $A(1,3)$, $B(5,9)$, $C(11, -1)$ form the triangle ABC.

(a) Find the coördinates of the midpoints of the sides of the triangle.

(b) Find the slopes of the sides of the triangle.

49. The equation of a straight line. The position of a straight line is determined when we know two points of the line.

Let a straight line be drawn through the points $A(2, 3)$ and $B(7, 5)$ and let $P(x, y)$ be any point anywhere on the line. Then, as we proceed from A to P, the slope is $\frac{y-3}{x-2}$,

and, as we proceed from A to B, the slope is $\frac{2}{5}$. Since the slope remains unchanged, $\dfrac{y - 3}{x - 2} = \frac{2}{5}$, or $5y - 2x = 11$.

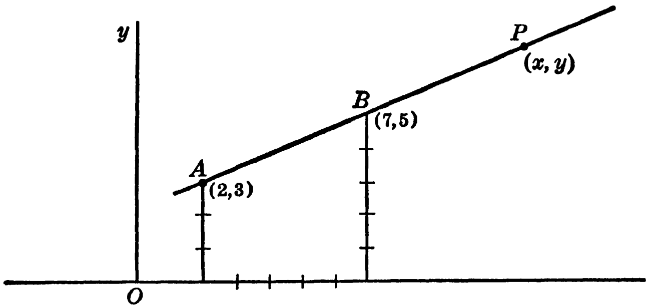


Fig. 19.

The equation shows how the x and y of any point P are related and is called the *equation of the line AB*. The equation $5y - 2x = 11$ enables us to find any number of additional points on the line by merely assigning different values to x and finding the corresponding value of y in each case. Thus if $x = 1$, $y = 2.6$ and $(1, 2.6)$ is a point on the line. To find whether or not a point $C(h, k)$ lies on AB or its extension, we substitute $x = h$ and $y = k$ in the equation $5y - 2x = 11$. The point C lies on the line AB if the result of the substitution is an identity and not otherwise. Thus $(42, 19)$ is on the line and $(4, 2)$ is not on the line.

The equation of the line enables us to determine just where the line AB crosses each of the coördinate axes. At the point where the line crosses the x axis, $y = 0$ and the equation becomes $0 - 2x = 11$. Hence $x = -5.5$. The distance from the origin to $x = -5.5$ is called the *x intercept*. Similarly, the y intercept is $+2.2$.

Take any line MN, Fig. 20, whose slope is s, and which crosses the y axis at $L(0, b)$, and let $P(x, y)$ be any point

on the line. When we move from L to P, the rise is BP, or $y - b$, and the advance is OA, or $x - 0$. Hence $\frac{y-b}{x-0} = s$,

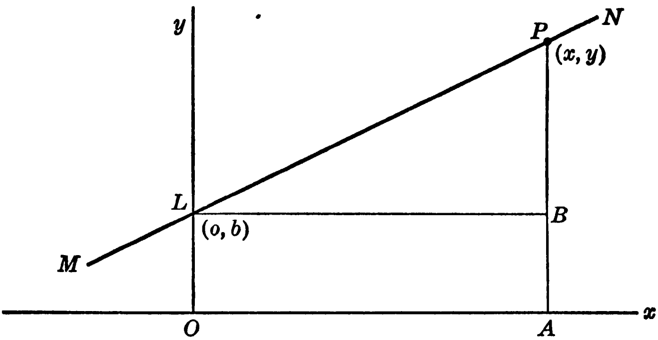


Fig. 20.

or $y = sx + b$, which is one of the standard forms of the equation of a straight line, called the *slope-intercept form.*

The equation $5y - 2x = 11$ may be written in the standard form by solving for y, the result being

$$y = \frac{2x}{5} + \frac{11}{5}.$$

Hence s, the slope, is $\frac{2}{5}$, and b, the y intercept, is $\frac{11}{5}$.

Any equation of the form $Ax + By + C = 0$, where A, B, C are definite numbers, can be written as $y = -\frac{Ax}{B} - \frac{C}{B}$, which is the same form as $y = sx + b$. Hence it represents a straight line, and $Ax + By + C = 0$ is the general equation of a straight line.

For some values of A, B, and C, the general equation may reduce to such forms as $x = 2$, $y = -3$, $x = 0$, $y = 0$. The equation $x = 2$ tells us that for all points on the line the value of x is 2. It is therefore a straight line perpen-

dicular to the x axis at a point 2 units to the right of the origin.

Exercise 39

1. State the slope and the intercept on the y axis for the lines:

(a) $y = 2x - 3$
(b) $y = \frac{2x}{3} + \frac{3}{5}$
(c) $3y = 4x - 5$
(d) $3x + 4y - 7 = 0$
(e) $\frac{3x}{2} = \frac{5y}{4} + 7$
(f) $x = 4y - 2$

2. Describe the positions of the lines:

(a) $3x = 5$
(b) $4y = 7$
(c) $x = 0$
(d) $y = 0$

3. Given the line $3x + 2y = 13$. Determine which of the following points are on the line:

(a) (2,3)
(b) (3,2)
(c) (7,−4)
(d) (−1,8)
(e) (0,0)
(f) (1,4)

4. Find the equation of the line that passes through the points:

(a) (2,5) and (4,8)
(b) (3,7) and (0,1)
(c) (−1,−2) and (−5,−7)
(d) (0,0) and (4,−3)

5. Given: A (2,3); B (4,8); C (7,12); D (5,7).

(a) Show that AB is parallel to DC and BC is parallel to AD, or that $ABCD$ is a parallelogram.

(b) Find the equations of the 6 lines determined by $ABCD$.

6. The points $\begin{array}{c|cccc} x & 1 & 5 & 9 & h \\ \hline y & k & 4 & 10 & 13 \end{array}$ are collinear—that is, they lie on a straight line. Find the values of h and k.

7. Plot the points A (5,4), B (9,10) and calculate the length of AB from a right triangle.

50. Linear interpolation. Tony walked north on 5th Ave., and the time was 12:10 at 34th St. and 12:28 at 58th St. That is, he walked a total distance of 24 blocks (equal in length) in 18 minutes. (a) When was he at

38th St.? (b) Where was he at 12:22? Assuming that he walked at a uniform pace, we may say:

(a) To reach 38th St. he walked 4 blocks, which is $\frac{1}{6}$ of the total distance of 24 blocks. Therefore he spent $\frac{1}{6}$ of the total time of 18 minutes, or 3 minutes, and reached 38th St. at 12:13.

(b) At 12:22 he had been walking 12 minutes, which is $\frac{2}{3}$ of the total time of 18 minutes. Therefore he had walked $\frac{2}{3}$ of the total distance of 24 blocks, or 16 blocks, and was at 50th St.

In general, if at x minutes past 12 Tony is at the street numbered y, he has walked $x - 10$ minutes, which is $\frac{x - 10}{18}$ of the total time, and has covered $y - 34$ blocks, which is $\frac{y - 34}{24}$ of the total distance. Since these fractions are equal if the pace is uniform,

$$\frac{x-10}{18} = \frac{y-34}{24},$$

or

$$\frac{y-34}{x-10} = \frac{24}{18} = \frac{4}{3}, \quad 3y - 4x = 62.$$

But this equation is represented graphically by a straight line that joins the points (10, 34) and (28, 58). That is, the assumption that the pace is uniform is the same as the assumption that for the table

x	10	a	22	28
y	34	38	b	58

the points (10, 34) and (28, 58) are joined by a straight line and that the points (a, 38) and (22, b) are on this line.

The process of finding $x = a$ when $y = 38$, between 34 and 58, or of finding $y = b$ when $x = 22$, between 10 and 28, is called *linear interpolation.*

a may be found from the equation $3y - 4x = 62$ by setting $y = 38$, and b by setting $x = 22$. But a simpler method is the method shown at the beginning of the discussion. That is, since 38 is $\frac{1}{6}$ of the way from 34 to 58, a must also be $\frac{1}{6}$ of the way from 10 to 28, or

$$a = 10 + \tfrac{1}{6}\,(28-10) = 13;$$

and, since 22 is $\frac{2}{3}$ of the way from 10 to 28, b must also be $\frac{2}{3}$ of the way from 34 to 58, or

$$b = 34 + \tfrac{2}{3}\,(58-34) = 50.$$

Suppose now that, while Tony was walking north, Walter was walking south along 5th Ave. and he was at 72nd St. at 12 o'clock and at 32nd St. at 12:25. Walter's position at 12:15, or the time when he reached 42nd St., may be found from

$$\begin{array}{c|cccc} x & 0 & 15 & d & 25 \\ \hline y & 72 & c & 42 & 32 \end{array}.$$

Since 15 is $\frac{15}{25}$ or $\frac{3}{5}$ of the way from 0 to 25, c is $\frac{3}{5}$ of the way from 72 to 32, and $c = 72 - \frac{3}{5}\,(72 - 32) = 48$.

Since 42 is $\frac{30}{40}$ or $\frac{3}{4}$ of the way from 72 to 32, d is $\frac{3}{4}$ of the way from 0 to 25, and $d = 0 + \frac{3}{4}\,(25 - 0) = 18\frac{3}{4}$.

That is, at 12:15 Walter was at 48th St. and reached 42nd St. at 12:18$\frac{3}{4}$.

Walter's position at any time may also be found from the equation of the straight line that joins the points (0, 72) and (25, 32), the equation being $5y + 8x = 360$.

If we wish to know when and where Tony and Walter passed each other, we need only solve the simultaneous equations:

$$\begin{aligned} 3y - 4x &= 62 \\ 5y + 8x &= 360, \end{aligned}$$

and find $x = 17\frac{1}{2}$, $y = 44$, and conclude that they passed each other at 44th St. at 12:17$\frac{1}{2}$.

Exercise 40

1. What part of the way from 6 to 18 are each of the numbers: 9, 10, 12, 15, 8.2, $7\frac{1}{2}$?

2. A number is a fractional part of the way, x, from 8 to 24. Find the number for $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{2}{3}, \frac{3}{8}, .7, .14\frac{2}{7}$.

3. A number x is $\frac{3}{5}$ of the way from 7 to y. Find y for $x = 10, 13, 7\frac{1}{2}$, 9.4, 32, 100.

4. Find what part of the way 6 is

(a) from 5 to 10;

(b) from $4\frac{1}{2}$ to $7\frac{1}{2}$;

(c) from 1 to $7\frac{3}{4}$;

(d) from $5\frac{1}{4}$ to 7.

5. Find by linear interpolation the values of h and k in the following:

(a)

x	3	h	20	27
y	8	11	k	36

(b)

x	16	h	9	6
y	3	6	k	11

(c)

x	2	h	3	4
y	100	95	k	11

(d)

x	-4	h	0	18
y	2	0	k	-8

(e)

x	2.718	h	3	3.142
y	4.973	5	k	8.647

6. In examples 5 (a), (b), (c), and (d), find the equation of the line through the two points whose coördinates are given in each case. Then find the values of h and k from the equation of the line.

7. Given A (3, 5) and B (7, 14). Find the coördinates of the point P, which lies between A and B on the line AB and which divides AB into two parts so that $\frac{AP}{AB}$ is (a) $\frac{1}{2}$; (b) $\frac{1}{3}$; (c) $\frac{2}{3}$; (d) $\frac{3}{8}$.

8. Given the points A (3, 5), B (5, 11), C (9, 1). Draw the triangle ABC and designate the midpoint of AB by L, of BC by M, and of AC by N.

(a) Find the coördinates of L, M, and N.

(b) Show that the slope of LM is equal to the slope of AC or that LM is parallel to AC; that NL is parallel to BC; that MN is parallel to AB.

(c) The point P is on AM, so that $\frac{AP}{AM} = \frac{2}{3}$. Find the coördinates of P.

(d) Show that the point P of example (c) is on CL, so that $\frac{CP}{CL} = \frac{2}{3}$, and on BN, so that $\frac{BP}{BN} = \frac{2}{3}$.

AM, BN, and CL are called the *medians* of the triangle, and their intersection point, P, is called the *center of gravity* of the triangle.

51. Use of linear interpolation. Mathematical calculations are often made with the aid of tables such as those of square root and cube root. Obviously a table cannot be made for all possible numbers, and in order to use any table effectively it is necessary to interpolate. Thus if we wish to find the value of $\sqrt[3]{403}$, we note that this number is not in the table but that it lies between two numbers that are in the table—namely, $\sqrt[3]{400} = 7.36806$ and $\sqrt[3]{410} = 7.42896$. We now write

x (number)	400	403	410
y (cube root)	7.36806	a	7.42896

Since 403 is $\frac{3}{10}$ of the way from 400 to 410,

$$a = 7.36806 + \tfrac{3}{10} \text{ of } (7.42896 - 7.36806) = 7.38633.$$

The value $a = \sqrt[3]{403} = 7.38633$ found by linear interpolation is not correct because it was calculated on the assumption that the points $(400, \sqrt[3]{400})$ and $(410, \sqrt[3]{410})$ are joined by a straight line and that the intermediate point $(403, \sqrt[3]{403})$ is on this line.

The graphic representation of the extended table of cube roots,

x	1	8	27	64	125
y	1	2	3	4	5,

consists of points that lie on a curve and not on a straight line. Therefore the result of the linear interpolation is an approximation and should not be regarded as a correct result.

We shall see later (page 177) how a more accurate value of $\sqrt[3]{403}$ may be found by another method of interpolation. For the present, however, it is to be understood that interpolation means linear or simple interpolation.

Illustration

The following entries are made from a certain table, and the **values of** a and b are required.

x	3%	$3\frac{1}{2}$%	4%	$3\frac{1}{3}$%	b
y	14.8775	14.2124	13.5903	a	14.0000

Since $3\frac{1}{3}\%$ is $\frac{2}{3}$ of the way from 3% to $3\frac{1}{2}\%$, a is $\frac{2}{3}$ of the way from 14.8775 to 14.2124, and $a = 14.8775 - \frac{2}{3}\,(14.8775 - 14.2124) = 14.4341$.

Since 14.0000 is $\frac{2124}{6221}$ of the way from 14.2124 to 13.5903, b is $\frac{2124}{6221}$ of the way from $3\frac{1}{2}\%$ to 4%, and $b = 3\frac{1}{2}\% + \frac{2124}{6221}$ of $\frac{1}{2}\% = 3.67\%$.

It is to be noted that if $3\frac{1}{3}\%$ is regarded as being $\frac{1}{3}$ of the way from 3% to 4% and a is found accordingly, $a = 14.4484$, a value that differs considerably from 14.4341.

The three pairs of corresponding given values of x and y are not on a line but on a curve. But the line that joins the first and second points is closer to the curve than the line that joins the first and third points. Therefore, 14.4341 is a better approximation of a than 14.4484.

That is, the result of linear interpolation is more nearly correct for a smaller interval of interpolation than for a larger interval.

Exercise 41

1. Use the table of square roots and find by linear interpolation the values of: (a) $\sqrt{623}$; (b) $\sqrt{62.3}$; (c) $\sqrt{48.6}$; (d) $\sqrt{486}$.

2. Use the table of cube roots and find the values of: (a) $\sqrt[3]{5}$, $\sqrt[3]{60}$, $\sqrt[3]{500}$; (b) $\sqrt[3]{.6}$, $\sqrt[3]{.62}$, $\sqrt[3]{.06}$.

3. Find by linear interpolation the values of: (a) $\sqrt[3]{547}$; (b) $\sqrt[3]{54.7}$; (c) $\sqrt[3]{5.47}$.

52. Linear equations. An equation of the form $ax + by + c = 0$ is called a *linear equation* because its graphic representation is a straight line. Two such linear equations are represented by two lines that either cross or are

parallel. The coördinates of the crossing point are values of x and y that fit both equations and constitute the solution of the simultaneous equations. Inconsistent equations such as $x + y = 6$ and $2x + 2y = 7$ are represented by parallel lines. Dependent equations such as $x + y = 6$ and $3x + 3y = 18$ are represented by two lines that coincide.

Exercise 42

1. Given the equations $3x - 4y = 12$ and $4x + 3y = 6$.

(a) Draw the graphs of two equations.

(b) Estimate the values of the coördinates of the point where the lines cross.

(c) Solve the simultaneous equations algebraically.

(d) Compare the graphic solution with the algebraic solution.

2. Given the equations $3x - 4y = 24$ and $6x - 8y = 7$.

(a) Draw the graphs of the two equations.

(b) Show algebraically that the slopes of the two lines are the same and that the lines are parallel.

3. (a) Through the origin, draw lines whose slopes are $+\frac{3}{2}$ and $-\frac{2}{3}$.

(b) Show that the lines are perpendicular to each other.

4. Given the equation $3x + 4y = 12$ and the point (5,6).

(a) Represent the given data graphically.

(b) Draw a line through the given point parallel to the given line.

(c) Write the equations of the parallel line.

(d) Draw a line through the origin perpendicular to the given line, and calculate the slope of the perpendicular.

(e) Draw a line through the given point perpendicular to the given line and calculate its slope.

53. Graphic solution of an algebraic equation. Suppose it is required to solve the equation $2x^3 - 15x + 5 = 0$. Write instead the two equations $y = 2x^3 - 15x + 5$ and $y = 0$. The solution of the given equation is the value or values of x of the points where the graphs of the two equations cross. But since $y = 0$ is the x axis, it is merely necessary to find where the graph of $y = 2x^3 - 15x + 5$

crosses the x axis. Before the graph can be drawn, a table of corresponding values of x and y must be prepared. An inspection of the table

x	-3	-2	-1	0	1	2	3
y	-4	19	18	5	-8	-9	14
Point	A	B	C	D	E	F	G

enables us to conclude that there is a crossing place between $x = -3$ and $x = -2$ since A is below the x axis and B is above the x axis. The other crossing places are between points D and E, and between points F and G. That is, the given equation has three roots, one between -3 and -2, another between 0 and 1, and the third between 2 and 3.

Approximate values of the roots may be found by linear interpolation. Thus the root between 2 and 3 is found from the table

x	2	h	3
y	-9	0	14

Since 0 is $\frac{9}{23}$ of the way from -9 to 14, h is $\frac{9}{23}$ of the way from 2 to 3 and $h = 2 + \frac{9}{23}$ of $1 = 2.4$.

Exercise 43

1. Find approximate values for the remaining roots of $2x^3 - 15x + 5 = 0$.

2. Draw the graph of $y = 2x^3 - 15x + 5$.

3. Find the approximate value of one root of $x^3 + 2x = 10$.

4. Draw the graph of $y = x^3 + 2x - 10$.

5. Draw the graphs of the equations $x^2 + y = 7$ and $y^2 + x = 4$.

6. Show that the graphic solution of the simultaneous equations of example 5 are approximately $(2.4,1.2)$, $(2.8,-1.2)$, $(-2.2,2.5)$, and $(-3.2,-2.6)$.

7. If two equations are given and their graphs do not cross, what statement can you make about the given equations?

8. Solve graphically the simultaneous equations $y = 3x^2 - 7x + 3$ and $2x + y = 6$.

9. Eliminate y from the equations of example 8 and solve the resulting quadratic equation for x to three decimal places.

54. The parabola. The graph of the equation $y = 3x^2 - 7x + 3$, Fig. 4, page 71, is called a *parabola*.

It is symmetrical about a vertical line drawn through its lowest point, the position of which may be calculated as follows: Solve the quadratic $3x^2 - 7x + 3 - y = 0$ and obtain

$$x = \frac{7 \pm \sqrt{7^2 - 4(3)\,(3-y)}}{2(3)} = \frac{7 \pm \sqrt{13 + 12y}}{6}$$

or

$$x = \frac{7 \pm \sqrt{12\,(y+\frac{13}{12})}}{6}$$

We now see that, for x to be real, $y + \frac{13}{12}$ may not be negative and y may not be less than $-\frac{13}{12}$. When $y = -\frac{13}{12}$,

$$x = \frac{7 \pm 0}{6} = \frac{7}{6}.$$

That is, the least value that y can have is $-\frac{13}{12}$, and this value arises when $x = \frac{7}{6}$. The lowest point of the parabola is at $x = \frac{7}{6}$, $y = -\frac{13}{12}$. From the method of obtaining the value of x, it appears that $x = \dfrac{-b}{2a} = +\frac{7}{6}$.

In general, if, in $y = ax^2 + bx + c$, and a is positive, the graph is a parabola having the same general appearance as the graph of $y = 3x^2 - 7x + 3$. The lowest point of the parabola is at

$$x = \frac{-b}{2a}$$

and

$$y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c = \frac{4ac - b^2}{4a}.$$

If a is negative as in $y = -3x^2 - 7x + 3$, the curve

bends in the opposite direction, and there is, not a lowest point, but a highest point, which is located at

$$x = \frac{-b}{2a} = -\frac{7}{6},$$

and

$$y = -3\left(\frac{-7}{6}\right)^2 - 7\left(\frac{-7}{6}\right) + 3 = +\frac{85}{12}.$$

Exercise 44

1. Find the coördinates of the lowest point or the highest point in each of the following parabolas:

(a) $y = x^2 - 6x - 1$
(b) $y = 1 + 6x - x^2$
(c) $y = 3x^2 - 8x - 1$
(d) $y = 1 + 8x - 3x^2$
(e) $y = 2x^2 - 5x - 3$
(f) $y = 3 + 5x - 2x^2$

2. (a) How do the graphs of $y = 2x^2 - 5x - 3$ and $x = 2y^2 - 5y - 3$ compare?

(b) Has $x = 2y^2 - 5y - 3$ a highest or a lowest point?

3. Given the equation $y = x^2 - 4x + 4$.

(a) Draw the graph.

(b) Find the coördinates of the lowest point.

(c) Note that, if $y = 0$, the equation is $(x - 2)(x - 2) = 0$, and the two roots are both $x = 2$. What is the graphic significance of this result?

4. Given the equation of the parabola, $y = -2x^2 + 8x + 3$.

(a) Find the highest point of the parabola.

(b) Find the coördinates of the points where the parabola crosses the x axis and the y axis.

(c) Find the equation of the line that is drawn through the highest point of the parabola and the point where the parabola crosses the y axis.

CHAPTER V

LOGARITHMS

55. Zero, negative, and fractional exponents. The primary conception of integral numbers arose from the necessity of representing the result of counting objects. The operations of addition and multiplication with integers still kept the field of numbers restricted to integers, but the inverse operations, division and subtraction, extended the field of numbers by introducing fractions, zero, and minus numbers.

Exponents have thus far been restricted to positive integers. The expression c^x has had a definite meaning only for positive integral values of the exponent x, whereas c may have been integral, fractional, positive, negative, or zero. In order to remove the restriction on the exponent x and to give meanings to such expressions as 5^0, 3^{-2}, $9^{1/2}$, $8^{-2/3}$, and so on, we shall assume that the law $a^x \cdot a^y = a^{x+y}$ holds not only for positive integral values of the exponents x and y, but for zero, negative, and fractional values as well.

(a) *Zero exponent.* Let $a^0 = x$ and multiply both members by a^5. Then $a^0 \cdot a^5 = xa^5$. But by our assumption, $a^0 \cdot a^5 = a^{0+5}$. Hence $a^5 = xa^5$ and $x = \dfrac{a^5}{a^5} = 1$. That is, $a^0 = 1$ for any value of a except 0. The symbol 0^0 remains undefined.

(b) *Negative exponent.* Let $a^{-4} = x$ and multiply both members by a^4. Then $a^{-4} \cdot a^4 = xa^4$. But by our assumption $a^{-4} \cdot a^4 = a^{-4+4} = a^0 = 1$. Hence $1 = xa^4$ and $x = \dfrac{1}{a^4}$.

That is, $a^{-4} = \frac{1}{a^4}$, and a^{-4} is the reciprocal of a^4.

In general, $a^{-n} = \frac{1}{a^n}$.

(c) *Fractional exponent.* Let $a^{1/3} = x$. Then $a^{1/3} \cdot a^{1/3} \cdot a^{1/3} = x \cdot x \cdot x$, or $a^1 = x^3$ and $x = \sqrt[3]{a}$, or $a^{1/3} = \sqrt[3]{a}$.

Let $a^{2/3} = y$. Then $a^{2/3} \cdot a^{2/3} \cdot a^{2/3} = y \cdot y \cdot y$, or $y^3 = a^2$ and $y = \sqrt[3]{a^2}$, or $a^{2/3} = \sqrt[3]{a^2}$.

Also, $a^{2/3} = a^{1/3} \cdot a^{1/3} = \sqrt[3]{a}\sqrt[3]{a} = (\sqrt[3]{a})^2$. Hence $a^{2/3} = \sqrt[3]{a^2}$ or $(\sqrt[3]{a})^2$.

In general, $a^{m/n} = \sqrt[n]{a^m}$ or $(\sqrt[n]{a})^m$.

The fractional power $\frac{1}{n}$ indicates the nth root; the fractional power $\frac{m}{n}$ indicates either the nth root of the mth power or the mth power of the nth root.

Illustrations

$(5.3)^0 = 1$; $\left(\frac{2}{3}\right)^{-2} = \frac{1}{(\frac{2}{3})^2} = \frac{9}{4}$ or $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$; $8^{-2/3} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{4}$; $5^{2/3} = \sqrt[3]{5^2} = \sqrt[3]{25}$.

The expression c^x now has a meaning for values of c and of x whether these values are positive or negative, integral or fractional. It is to be noted, however, that since $c^{m/n} = (\sqrt[n]{c})^m$, a negative value of c coupled with an even-numbered value of n results in an imaginary number.

Exercise 45

1. Express the following as ordinary arithmetic numbers:

$(\frac{2}{3})^3$, $(3.27)^0$, 6^{-3}, $(\frac{1}{4})^{1/2}$, $8^{1/3}$, $16^{-3/4}$, $(\frac{1}{8})^{-1/3}$, $9^{-3/2}$, $(-8)^{-2/3}$

2. Find the value of $8^{-2/3} \times (\frac{1}{4})^{-2} + 12^0 \div (\frac{4}{9})^{-1/2} - (\frac{9}{25})^{-1/2}$

3. Find the value of $x^{m/n}$ if:

(a) $x = 9$, $m = 3$, $n = 2$
(b) $x = 4$, $m = 3$, $n = -2$

(c) $x = -8,\ m = -2,\ n = -3$
(d) $x = \frac{4}{9},\ m = -4,\ n = 2$
(e) $x = \frac{4}{9},\ m = 0,\ n = -5$
(f) $x = \frac{2}{3},\ m = 4,\ n = -2$

4. Write the following without using fractional or negative exponents. Simplify when it is possible to do so.

(a) $x^{1/2} + x^{1/3}x^{1/6}$
(b) $x^{1/2}x^{1/3} - x^{-2}$
(c) $x^{1/2}x^{-1/3} - (x^{1/2})^{1/3}$
(d) $(x^2)^{-3}\,(x^{-3})^{-2}$
(e) $3x^{-2} + 2x^{-1} + (4x^2)^{1/2}$
(f) $(x^{-1} + y^{-1}) \div (x^{-1} - y^{-1})$

5. Note the following relations:

(a) $\sqrt{2}\ \sqrt{3} = 2^{1/2}\,3^{1/2} = (2 \times 3)^{1/2} = \sqrt{6}$
(b) $\sqrt{50} = (25 \times 2)^{1/2} = 25^{1/2} \times 2^{1/2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
(c) $\dfrac{\sqrt{3}}{\sqrt{2}} = \dfrac{\sqrt{3}\,\sqrt{2}}{\sqrt{2}\,\sqrt{2}} = \dfrac{\sqrt{6}}{2}$

In (a), the calculation of $\sqrt{6}$ requires less work than the calculation of $\sqrt{2} \times \sqrt{3}$. In (c), less calculation is required for $\dfrac{\sqrt{6}}{2}$ than for $\dfrac{\sqrt{3}}{\sqrt{2}}$. The form shown in (b) is useful for the simplification of such expressions as:

$$\sqrt{18} - \sqrt{50} + \sqrt{32}$$

$$\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2};\ \sqrt{50} = 5\sqrt{2};\ \sqrt{32} = 4\sqrt{2}$$

Hence $\sqrt{18} - \sqrt{50} + \sqrt{32} = 3\sqrt{2} - 5\sqrt{2} + 4\sqrt{2} = 2\sqrt{2}$.

6. Calculate the value of each of the following to 3 decimal places:

(a) $\sqrt{5}\sqrt{7}$; (b) $\dfrac{\sqrt{5}}{\sqrt{7}}$; (c) $\sqrt{45} + \sqrt{20} - \sqrt{80}$.

56. Logarithms. From the equation $y = 4^x$, the value of y is easily calculated for positive or negative integral values of x and also for some fractional values. But if $y = 9$, the equation is $4^x = 9$, and the value of x can be expressed only by introducing a new notation. In $y = 4^x$,

x is called the *logarithm* of y to the *base* 4 and is written

$$x = \log_4 y$$

In general, if $y = a^x$, $x = \log_a y$.

The logarithm of a number is the exponent that a selected base must have in order to produce that number. Thus

$$\log_3 81 = 4, \text{ because } 3^4 = 81.$$
$$\log_9 3 = \tfrac{1}{2}, \text{ because } 9^{1/2} = 3.$$
$$\log_5(\tfrac{1}{25}) = -2 \text{ because } 5^{-2} = \tfrac{1}{25}.$$

The two equations $y = a^x$ and $\log_a y = x$ express the same relation between x and y in different forms, and these two forms are interchangeable, one being called the *inverse* of the other.

Exercise 46

1. Express the following equations by means of logarithms:

(a) $19 = 10^x$
(b) $27 = 3^x$
(c) $(0.5)^x = 4$
(d) $4^x = 0.5$
(e) $27^x = 3$
(f) $(.25)^x = 64$

2. Express the following equations without using logarithms:

(a) $y = \log_2 3$
(b) $x = \log_3 2$
(c) $\log_4 10 = x$
(d) $\log_{10} 4 = y$
(e) $\log_5 25 = 2$
(f) $\log_3 243 = 5$

57. Logarithm tables. Logarithms may be calculated to any base except 0 or 1, but only two bases are ever used in practice, namely:

10, the base of our decimal systems of writing numbers, and e, a number that arises from $\left(1 + \frac{1}{z}\right)^z$ when z increases beyond all limits. The value of e is 2.718 . . . .

The logarithms of numbers are calculated and arranged in an orderly form, called a *table of logarithms*. Logarithms to the base e are called *Napierian* or *natural logarithms* and are used almost exclusively in higher mathematics. Loga-

rithms to the base 10 are called *Briggsian* or *common logarithms* and are always used for ordinary computations.

In subsequent discussions it will be understood that the base is 10, so that log 7 means $\log_{10} 7$.

58. Characteristic and mantissa. From the corresponding values,

10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
.001	.01	.1	1	10	100	1000

we may write $\log 1000 = 3$, $\log 1 = 0$, $\log .01 = -2$, and so forth.

The number 68.27 is between 10 and 100 or between 10^1 and 10^2 and may be written 10^{1+k} where k is a positive decimal fraction. The same sequence of digits gives such numbers as 682.7, .6827, .006827, and so on. These numbers are between 10^2 and 10^3, 10^0 and 10^{-1}, 10^{-3} and 10^{-2}, respectively, and we may write:

$$\begin{aligned} 68.27 &= 10^{1+k}, \text{ or } \log 68.27 = 1 + k, \\ 682.7 &= 10^{2+l}, \text{ or } \log 682.7 = 2 + l, \\ .6827 &= 10^{-1+m}, \text{ or } \log .6827 = -1 + m, \\ .006827 &= 10^{-3+n}, \text{ or } \log. 006827 = -3 + n, \end{aligned}$$

where k, l, m, n are positive decimal fractions. Then,

$$\begin{aligned} 10^{2+l} &= 682.7 = 10 \times 68.27 = 10^1 \times 10^{1+k} = 10^{2+k}, \\ 10^{1+k} &= 68.27 = 10^2 \times .6827 = 10^2 \times 10^{-1+m} = 10^{1+m}, \\ 10^{2+l} &= 682.7 = 10^5 \times .006827 = 10^5 \times 10^{-3+n} = 10^{2+n}. \end{aligned}$$

Hence $k = l = m = n$. That is, the decimal portions of the logarithms are identical for the same sequence of digits regardless of the position of the decimal point. The logarithm of a number is therefore separated into two parts—namely: (a) a positive or a negative integer called the *characteristic*, and (b) a positive decimal called the *mantissa*.

The numerical value of the characteristic depends upon the position of the decimal point in the given number, and the numerical value of the mantissa depends upon the sequence of the digits in the number.

Thus log 68.27 and log 6827 have the same mantissa and different characteristics; log 68.27 and log 72.86 have the same characteristic and different mantissas.

The logarithm of a number may appear in a form such as -2.4327, which means $-2 - .4327$, but this form is easily changed to $-3 + .5673$ and is usually written $\bar{3}.5673$. The minus sign written above the 3 indicates that only the characteristic is negative.

Exercise 47

1. The logarithms of certain numbers appear as $\bar{1}.3427$, $\bar{2}.8346$, $\bar{3}.4723$, $\bar{4}.2954$. Change these forms to ordinary negative numbers.

2. The logarithms of certain numbers appear as -0.6283, -1.2465, -2.3627, -3.0284. Change these forms so that they appear with negative characteristics and positive mantissas.

59. Characteristic. The following relations:

$$\begin{aligned} 0.6827 &= 10^{-1+k}, \\ 6.827 &= 10^{0+k}, \\ 68.27 &= 10^{1+k} \\ 682.7 &= 10^{2+k}, \end{aligned}$$

show that, for every step that the decimal point is moved toward the right, the characteristic is increased by 1, and that, for every step that the decimal point is moved toward the left, the characteristic is decreased by 1. Therefore, in order to determine the characteristic of the logarithm of any number, assume that the decimal point was initially after the first significant digit in the number and, hence, that the characteristic was 0. Count the number of places that the decimal point must be moved from the assumed position to the position that it actually occupies. The count gives the numerical value of the characteristic. The sign of the characteristic is plus if the motion is toward the right, and minus if the motion is toward the left.

Thus, to find the characteristic of log 682700, assume that

the decimal point was initially after the first significant digit, 6. The number was then between 1 and 10, and the characteristic of its logarithm was 0. For the decimal point to be brought to its actual position, it must be moved 5 places toward the right. The characteristic is therefore +5.

To find the characteristic of log 0.0006827, assume that the decimal point was initially after the first significant digit, 6. For the decimal point to be brought to its actual position, it must be moved 4 places toward the left. The characteristic is therefore -4, or $\bar{4}$.

60. Mantissa. A table of common logarithms is a table of positive mantissas. In Table II, the decimal point does not appear, but it is to be understood that, in the column headed *N*, the decimal point is between the two figures, and in the body of the table the decimal point is before the first figure.

The table gives the mantissas of the logarithms of numbers from 100 to 999, numbers of 3 figures, of which the first two figures are in column *N* and the third figure is at the head of one of the columns marked 0, 1, 2, . . . 9.

Illustrations

1. To find the mantissa of log 153, look in column *N* for the first two figures, 15. Then look along the horizontal line where the number 15 appears, until the column headed by the next figure of the number, 3, is reached. The decimal .1847 found at this place is the mantissa of log 153 or of log 1.53 or of log 0.00153.

2. To find the mantissa of log 15, write the number in the form 15.0 and look for the mantissa of log 150.

3. To find the mantissa of log 1534, note that 1534 is between 1530 and 1540. Find the mantissas of log 1530 and of log 1540. The following table results:

Number	*Mantissa*
1530	.1847
1534	m
1540	.1875

The missing mantissa, m, found by linear interpolation, is

$$m = .1847 + .4 \text{ of } (.1875 - .1847) = .1858.$$

61. Antilogarithm. The number that corresponds to a given logarithm is called the *antilogarithm* and is written *antilog*. Antilog $2.1866 = x$ is the same statement as $\log x = 2.1866$.

To find antilog 2.1866—that is, the number whose logarithm is 2.1866—disregard the characteristic 2 and look for the mantissa .1866 in the table. The mantissa .1866 is not in the table, but is between .1847 and .1875. We now have:

Number	*Mantissa*
1530	.1847
x	.1866
1540	.1875

The missing number, x, found by linear interpolation, is

$$x = 1530 + \frac{19}{28} \text{ of } (1540 - 1530) = 1537.$$

Now consider the characteristic 2 that was disregarded. Begin by placing the decimal point after the digit 1, in which case the characteristic would be 0. Since the characteristic is 2, move two places toward the right and place the decimal point after the digit 3, or $x = 153.7$. Thus:

$$\begin{aligned} \text{antilog } 3.1866 &= 1537; \\ \text{“} \quad \bar{1}.1866 &= 0.1537; \\ \text{“} \quad \bar{3}.1866 &= 0.001537. \end{aligned}$$

Caution: The logarithm of a number is an ordinary number whether it appears in the unusual form $\bar{3}.1866$ or in the usual form -2.8134. Only the first form is found

from the table. To perform the indicated multiplication $\bar{2}.6342 \times \bar{1}.4764$, the product is rewritten as $-1.3658 \times -.5236$.

Exercise 48

Fill in the blanks in the following tabulation to four significant digits for numbers and to four decimals for logarithms:

	Number	*Log*		*Number*	*Log*
1.	1614	_____	**11.**	6.6	_____
2.	0.001614	_____	**12.**	0.66	_____
3.	_____	1.2214	**13.**	0.0066	_____
4.	_____	$\bar{1}.2214$	**14.**	5.	_____
5.	191,300	_____	**15.**	5,673,000	_____
6.	19.13	_____	**16.**	_____	2.8270
7.	_____	−0.2958	**17.**	_____	$\bar{3}.8270$
8.	_____	3.2958	**18.**	1.988	_____
9.	_____	5.2958	**19.**	_____	1.988
10.	_____	$\bar{4}.2958$	**20.**	_____	−0.1

21. Find the values of: (a) $\log 2 \times \log 3$; (b) $\log 3 \div \log 2$; (c) $\log .2 \times \log .3$; (d) $\log .3 \div \log .2$.

62. Multiplication by logarithms.

$$\text{If } 10^a = x, \text{ then } \log x = a.$$
$$\text{If } 10^b = y, \quad \text{“} \quad \log y = b.$$

Multiplication gives

$$10^a \times 10^b = xy, \text{ or } 10^{a+b} = xy.$$

Hence

$$\log (xy) = a + b = \log x + \log y$$

Rule: *The logarithm of the product of two numbers is found by adding the logarithms of the factors.*

Thus, if

$$15.36 \times 2.743 = a,$$

$$\log a = \log 15.36 + \log 2.743$$
$$\log a = 1.1864 + 0.4383$$

$$\log a = 1.6247$$
$$a = 42.14.$$

The rule is applicable to any number of factors. Thus:

$$12.37 \times 0.04674 \times 8.923 \times 0.5836 = x;$$

$$\log x = \log 12.37 + \log 0.04674 + \log 8.923 + \log 0.5836$$

$$\begin{aligned} \log 12.37 &= 1.0924 \\ \log 0.04674 &= \bar{2}.6697 \\ \log 8.923 &= 0.9506 \\ \log 0.5836 &= \underline{\bar{1}.7661} \\ \log x &= 0.4788 \\ x &= 3.011. \end{aligned}$$

Note that the mantissas are added as in ordinary arithmetic addition, and there is 2 to carry. The carried 2 and the characteristics are added algebraically; that is, the sum of 2, 1, $\bar{2}$, 0, $\bar{1}$ is 0.

Exercise 49

Calculate by the use of logarithms:

1. 34.63×17.67
2. $1.732 \times 1.414 \times 2.449$
3. 0.00462×0.0583
4. 0.09642×5837
5. $17.46 \times 0.0837 \times .0025$
6. $75 \times 0.64 \times 3.75$
7. $0.58 \times 0.41 \times 0.76$
8. $x = \log .2 \times \log 3$
9. $x = \log 0.2 \times \log 5$
10. $x = \log 2.5 \times \log 4$

63. Division by logarithms. If $10^a = x$ and $10^b = y$, division gives

$$\frac{10^a}{10^b} = \frac{x}{y} \quad \text{or } 10^{a-b} = \frac{x}{y}.$$

Hence

$$\log \frac{x}{y} = a - b = \log x - \log y.$$

Rule: *The logarithm of the quotient of two numbers is found by subtracting the logarithm of the divisor from the logarithm of the dividend.*

Illustrations

1. $65.36 \div 2.743 = a$

$$\begin{aligned} \log a &= \log 65.36 - \log 2.743 \\ \log a &= 1.8153 - 0.4383 \\ \log a &= 1.3770 \\ a &= 23.83. \end{aligned}$$

2. $2.743 \div 65.36 = a$

$$\begin{aligned} \log 2.743 &= 0.4383 \\ \log 65.36 &= \underline{1.8153} \end{aligned}$$

Subtraction gives,

$$\begin{aligned} \log a &= \bar{2}.6230 \\ a &= 0.04198. \end{aligned}$$

Note in Illustration 2 that we subtract the mantissa of 6536 from the mantissa of 2743 in the ordinary way until we reach the characteristics. Then, since 1 was borrowed, the final step is to subtract 1 from -1. We may perform the subtraction and at the same time prove its correctness by adding the difference to the subtrahend. Then when we reach the characteristic, there is 1 to carry, and we ask $1 + 1 + ? = 0$. The missing number, -2, is the characteristic.

3. $\dfrac{0.04783 \times 59.46 \times 3727}{0.1843 \times 156.2 \times 89{,}620} = \dfrac{a}{b} = x$

$$\log x = \log a - \log b$$

$$\begin{array}{rlcrl} \log 0.04783 &= \bar{2}.6797 & \qquad & \log 0.1843 &= \bar{1}.2655 \\ \log 59.46 &= 1.7742 & & \log 156.2 &= 2.1937 \\ \log 3727 &= \underline{3.5713} & & \log 89{,}620 &= \underline{4.9524} \\ \log a &= 4.0252 & & \log b &= 6.4116 \\ \log b &= \underline{6.4116} & & & \\ \log x &= \bar{3}.6136 & & & \\ x &= 0.004108. & & & \end{array}$$

Exercise 50

Calculate by the use of logarithms:

1. $5835 \div 236$
2. $236 \div 5835$
3. $(6.75 \times 5.896) \div (2.642 \times 7.483)$
4. $x = \log 3 \div \log 9$

5. $x = \log 2 \div \log 3$
6. $x = \log 0.8 \div \log 0.4$
7. $x = (\log 2 \times \log 3) \div \log 6$
8. $(0.0438 \times 0.2647) \div (5.623 \times 0.0083)$

64. Powers by logarithms. If n is a positive integer,

$$\begin{aligned}\log (a^n) &= \log (a \cdot a \cdot a \ldots\ldots\ n \text{ factors})\\ &= \log a + \log a + \log a + \ldots\ldots\ n \text{ terms}\\ &= n \log a.\end{aligned}$$

Rule: *The logarithm of* $\mathrm{a^n}$ *if* n *is a positive integer is found by multiplying the logarithm of* a *by the exponent,* n.

Illustration

$$\begin{aligned}(0.6837)^5 &= x\\ \log x &= 5 \times \log 0.6837\\ \log x &= 5 \times \bar{1}.8349\\ \log x &= \bar{1}.1745\\ x &= 0.1494.\end{aligned}$$

The multiplication of $\bar{1}.8349$ by 5 was performed as follows: Multiply the mantissa by 5 in the ordinary way, and there is 4 to carry. Then say $5 \times (-1) = -5$, and the carried 4 makes -1.

Exercise 51

Calculate by the use of logarithms:

1. $(2.375)^2$
2. $(0.8375)^5$
3. $(1.045)^7$
4. $x = (\log 2)^2$
5. $x = (\log 3)^3$
6. $(0.375)^4$

65. Roots by logarithms. If in a^n , n is a fraction of the form p/q where p and q are positive integers, we write:

$$a^n = a^{p/q} = \sqrt[q]{a^p} = x.$$

Raise both members of $\sqrt[q]{a^p} = x$ to the power q and obtain

$$a^p = x^q.$$

Then

$$\log (a^p) = \log (x^q),$$

or

$$p \log a = q \log x,$$

or

$$(p/q) \log a = \log x.$$

Then, since p/q stands for n, and x stands for a^n, the last equation becomes:

$$n \log a = \log (a^n).$$

That is, the rule that was stated in the preceding article for exponents that are positive integers applies equally well if the exponents are fractions.

Illustrations

1. $\sqrt[3]{0.06743} = x = (0.06743)^{1/3}$.

$$\begin{aligned} \log x &= \tfrac{1}{3} \times \log (0.06743) \\ &= \tfrac{1}{3} \times \bar{2}.8289. \end{aligned}$$

To multiply $\bar{2}.8289$ by $\frac{1}{3}$, we may proceed as follows:

$$\begin{aligned} \bar{2}.8289 &= -2 + 0.8289 = -1.1711 \\ \tfrac{1}{3} \times -1.1711 &= -0.3904 = -1 + 0.6096. \end{aligned}$$

Hence

$$\begin{aligned} \log x &= \bar{1}.6096 \\ x &= 0.4070. \end{aligned}$$

We may also calculate $\frac{1}{3} \times \bar{2}.8289$ as follows:

$\bar{2}.8289$ may be written $\bar{3} + 1 + 0.8289$, or $\bar{3} + 1.8289$

$$\tfrac{1}{3} \times (\bar{3} + 1.8289) = \bar{1} + 0.6096 = \bar{1}.6096.$$

2. $(0.375)^{-0.375} = x$.

$$\begin{aligned} \log x &= -0.375 \times \log (0.375) \\ \log x &= (-0.375) \times (\bar{1}.5740) \\ \log x &= (-0.375) \times (-0.4260) \\ \log x &= 0.1598 \\ x &= 1.445. \end{aligned}$$

Exercise 52

Calculate by the use of logarithms:

1. $\sqrt{10}$
2. $\sqrt[3]{10}$
3. $\sqrt{2} \times \sqrt{3}$
4. $\sqrt{6}$
5. $\sqrt[3]{0.8}$
6. $\sqrt[5]{6.623}$
7. $\sqrt{\log 2}$
8. $9^{1/3}$
9. $\sqrt[3]{5} \times 1.7^2$
10. $(\sqrt[3]{7} \times 1.3^4) + (\sqrt[4]{1.3} \times 7^3)$

66. Exponential equations. The unknown number in an equation may appear in the exponent, as, for example, in the equation $4^x = 8$. It is sometimes possible to express such an equation, called an *exponential equation,* in a form that can be solved by inspection. The equation $4^x = 8$ may be written $(2^2)^x = 2^3$, or $2^{2x} = 2^3$, from which we conclude that $2x = 3$, or $x = 1.5$.

Equations of this type can always be solved by logarithms as follows:

If
$$4^x = 8,$$
then
$$\log (4^x) = \log (8)$$
$$x \log 4 = \log 8$$
$$x(0.6021) = 0.9031$$
$$x = \frac{0.9031}{0.6021} = 1.49991.$$

But since the dividend and the divisor of $\frac{9031}{6021}$ are accurate to four digits, the quotient is accurate to four digits only, and the result, 1.49991, must be expressed as 1.500.

Exercise 53

Find the value of x in each of the following equations by logarithms. Wherever it is possible to do so, also find x without using logarithms.

1. $(1.25)^x = 20$
2. $9^x = 27$
3. $(2.7)^x = 27$
4. $\log x = 0.5$
5. $(1 + x)^5 = 20$
6. $4^{-2x} = 8$
7. $(1.5)^{3x} = 2.25$
8. $(2.25)^{3x} = 1.5$
9. $(2.5)^x = 0.064$
10. $(0.83)^x = 0.125$

67. Arrangement of work. Time may be saved by arranging the work for calculations by logarithms so as to avoid needless repetitions. In the following illustration, the arrangement of the work is indicated.

Illustration

$$\frac{\sqrt[3]{.2783} \times (.4727)^2}{\sqrt{.5963} \times (.5263)^3} = n.$$

Rewrite the roots as fractional exponents and take the logarithms of both sides of the equation. Then

$$\begin{aligned}\log n &= \log\,(.2783^{1/3} \times .4727^2) - \log\,(.5963^{1/2} \times .5263^3)\\ &= [\log .2783^{1/3} + \log .4727^2] - [\log .5963^{1/2} + \log .5263^3]\\ &= [\tfrac{1}{3}\log .2783 + 2 \log .4727] - [\tfrac{1}{2}\log .5963 + 3 \log .5263].\end{aligned}$$

The work may be arranged as in the following tabulation without setting down the steps shown above.

Number	*Log*	*Mult. by*	*Result*
.2783	______	$\frac{1}{3}$	______
.4727	______	2	______
.5963	______	$\frac{1}{2}$	______
.5263	______	3	______
$\log n =$	______		
$n =$	______,	ANSWER.	

The student may complete the calculation.

Exercise 54

Find the value of x in each of the following:

1. $x = 1.674 \times .04784$
2. $x = 1.674 \div .04784$
3. $x = .04784 \div 1.674$
4. $x = 1.05^{20}$
5. $x = .95^{20}$
6. $1.06^x = 2$
7. $(1 + x)^{20} = 3$
8. $x = \sqrt{.7547}$
9. $x = \sqrt[3]{.7547}$
10. $x = 10^{2.3782}$
11. $x = \dfrac{672 \times 387 \times 463}{596 \times 439 \times 784}$
12. $x = \sqrt[3]{87.42} \times \sqrt[5]{.0367}$
13. $x = (.875)^{-1.5}$
14. $x = (1.875)^{-1.5}$

68. Forms of equations. By means of the rules

(1) $\log\,(a \times b) = \log a + \log b,$
(2) $\log\,(a \div b) = \log a - \log b,$
(3) $\log\,(a^n) = n \cdot \log a,$

it is often possible to change the form of an equation that involves logarithms to another form that does not.

Illustrations

1.

$$3 \log x - 2 \log y = 4$$

takes the successive forms

$$\log (x^3) - \log (y^2) = 4$$

$$\log \frac{(x^3)}{(y^2)} = 4$$

$$\frac{x^3}{y^2} = 10^4$$

$$x^3 = 10^4 y^2.$$

The original and the final equations both show how x and y are related, but the graph can be constructed more easily from $x^3 = 10^4 y^2$.

2. $x = 3^{\log 2/\log 3}$ is merely another form of the equation $x = 2$. Take logarithms of both members, $\log x = \dfrac{\log 2}{\log 3} \times \log 3 = \log 2$. Then, since $\log x = \log 2$, x and 2 have the same logarithm and therefore represent the same number, and $x = 2$.

Exercise 55

Write each of the following equations in a form that does not contain logarithms:

1. $3 \log x = \log 8$

2. $3 \log x - \log 8 = 27$

3. $2 \log x + 3 \log y = 1$

4. $\frac{1}{2} \log x + \frac{1}{3} \log y = \log 3$

5. $(\log 8) \div (\log 2) = \log (x^n) \div \log x$

6. $x = 3^{\log 5/\log 3}$

7. $\log y = 2 \log x - 1$

8. $\log y = 3x + 1$

69. Use of short tables. If only the following logarithms are known:

$$\log 2 = 0.3010$$
$$\log 3 = 0.4771$$
$$\log 7 = 0.8451$$
$$\log 11 = 1.0414,$$

it is possible to calculate the logarithm of a number that can be expressed in terms of the factors 2, 3, 7, 11.

Illustrations

1. Find $\log x$ from $x = (1.05)^{20}$ by using the short table.

$$\begin{aligned} \log x &= 20 \log \frac{3 \times 7}{2 \times 10} \\ \log x &= 20(\log 3 + \log 7 - \log 2 - \log 10) \\ \log x &= 0.4240. \end{aligned}$$

2. Find the value of log 4620 by using the short table.

Since $\quad 4620 = 2 \times 3 \times 7 \times 10 \times 11,$

$$\begin{aligned} \log 4620 &= \log 2 + \log 3 + \log 7 + \log 10 + \log 11 \\ &= 3.6646. \end{aligned}$$

Exercise 56

Without referring to a table of logarithms, but using the values of log 2, log 3, log 7, and log 11, calculate x from each of the following equations:

1. $x = \log 11.2$

2. $x = \log 350$

3. $x = \log 4.2$

4. $x = \log 64.8$

5. $x = \log (3\frac{3}{7})$

6. $3^x = 21$

7. $(1.5)^x = 3.5$

8. $(2.25)^x = 10$

9. $(1.08)^x = 6.25$

10. $10^{3x} = 200$

11. $x = \log 132$

12. $x = \log 3\frac{1}{7}$

13. $(3\frac{1}{7})^x = 4.4$

14. $(1\frac{2}{9})^x = \frac{8}{7}$

15. $(\frac{8}{7})^x = 1\frac{2}{9}$

16. Determine the number of digits in each of the following numbers and estimate the first digit:

(a) 9^9; (b) 2^{50}; (c) 1.05^{100}; (d) 3^{3^3}.

17. Draw the graph of $y = 2^x$ for $x = -3, -2, -1, 0, 1, 2, 3$.

18. Draw the graph of $y = \log x$ for $x = .1, 1, 2, 3, 4, 5, \ldots 10$.

PART II

FINANCIAL MATHEMATICS

CHAPTER VI

COMPOUND INTEREST AND ANNUITIES

70. Compound interest. People save money for various purposes, and the savings can produce income by being loaned to responsible individuals or companies that need money to carry on or to expand business enterprises. The savings are then said to be invested, and the investment, the amount loaned, is called *principal.* The borrower agrees to pay interest for the use of the principal and specifies at what rate the interest shall be, how frequently it is to be paid, and when the principal shall be repaid.

Thus, Jones lends \$1000 to Green, and they agree that the principal, \$1000, shall be repaid at the end of 3 years, and that interest shall be paid annually at 4% per annum. The agreement means that the interest for a year is 4% of \$1000, or \$40, and that Green will pay \$40 at the end of the first, second, and third years and also \$1000 at the end of 3 years.

It may be specified that interest shall be paid semiannually at 4% per annum. It is then understood by both Jones and Green that each semiannual interest payment will be $\frac{1}{2}$ of \$40, or \$20, and that 6 such interest payments will be made. Similarly, an agreement that interest shall be paid quarterly at 4% per annum is understood to mean that an interest payment of \$10 will be made at the end of every 3-month period.

When Jones receives an interest payment of \$40, \$20, or \$10, it becomes available for investment and constitutes additional principal which can also earn interest. Jones and Green may agree that each interest payment as it

becomes due shall be retained by Green as an additional loan on which he is to pay interest at the same rate as on the original loan. If the agreement was that interest is payable semiannually, then at the end of 3 years the account is:

Principal at the beginning	\$1000.000
Interest for 6 months, 2% of \$1000	20.000
Principal at the end of 6 months	1020.000
Interest for 6 months	20.400
Principal at the end of 1 year	1040.400
Interest for 6 months	20.808
Principal at the end of $1\frac{1}{2}$ years	1061.208
Interest for 6 months	21.224
Principal at the end of 2 years	1082.432
Interest for 6 months	21.649
Principal at the end of $2\frac{1}{2}$ years	1104.081
Interest for 6 months	22.082
Principal at the end of 3 years	1126.163

That is, if interest is converted into principal semiannually at 4% per annum, the total amount (principal and accumulated interest) that Green should pay at the end of 3 years for a loan of \$1000 is \$1126.16, the interest payment being \$126.16.

Interest calculated by assuming that the principal changes periodically because interest is converted into principal is called *compound interest*. In all investment calculations it is assumed that money is never idle and that when interest is paid it is reinvested immediately. Therefore the investment earns compound interest.

The interval of time for which each interest payment is specified is called an *interest period* or merely a *period*. The phrases—interest at 1% per quarterly period, interest payable quarterly at 4% per annum, interest at 4% per annum compounded or convertible quarterly—have the

same meaning, namely, that the principal is increased by 1% at the end of each period of 3 months.

Exercise 57

State the length of a period, the number of periods, and the rate per period in each of the following:

1. \$1000 is invested for 5 years, and interest at 6% per annum is compounded (a) annually; (b) semiannually; (c) quarterly; (d) monthly.

2. \$5000 is invested for $4\frac{1}{2}$ years, and interest at 4% per annum is convertible (a) semiannually; (b) quarterly; (c) monthly.

3. \$100 is invested for 3 years, and interest at $4\frac{1}{2}$% per annum is payable (a) semiannually; (b) quarterly; (c) monthly.

4. \$500 is invested for $5\frac{1}{2}$ years, and interest at 5% per annum is compounded (a) semiannually; (b) quarterly; (c) monthly.

5. \$1000 is invested for $8\frac{1}{2}$ years and interest at 3% per annum is convertible (a) semiannually; (b) quarterly; (c) monthly.

71. Compound interest symbols. In Fig. 21, the marks indicated by the letters A, B, C, D, . . . are equally spaced and represent dates at equal intervals of time. The equal spaces between the division points represent interest periods. (It may be more convenient to draw the line AR horizontally.) If p is invested at date A at rate i per period, its value at date B is $p\ (1 + i)$. That is, the value of an investment of p after the lapse of one interest period is $(1 + i)$ times as much as it was at the beginning of the period. The value at date C of the investment of p made at date A is the same as the value at date C of the investment $p\ (1 + i)$ made at date B and is

Fig. 21.

$$p\ (1+i) + ip\ (1+i) = p\ (1+i)\ (1+i) = p\ (1+i)^2$$

At date D the value is

$$p\ (1+i)^2 + ip\ (1+i)^2 = p(1+i)^2\ (1+i) = p\ (1+i)^3,$$

and so on.

At the end of n periods, at date R, the value of the investment of p made at date A is $p\ (1 + i)^n$. The expression $(1 + i)^n$ is represented by the symbol s^n_i (ess-en-eye) and indicates the total value, principal and interest, of an investment of \$1 at the end of n periods, if interest is compounded at rate i per period.

That is,

$$s^n_i = (1 + i)^n$$

and

$$ps^n_i = p\ (1 + i)^n.$$

Thus $100s^5_{1\%}$ may be stated in words as follows: If I owe \$100 payable at date A and defer the payment to date F, 5 periods later, the amount that I should pay at date F is indicated by $100s^5_{1\%}$, provided interest is compounded at 1% per period.

On the other hand, if I owe \$100 payable at date F, the payment that will cancel the debt at date A, 5 periods before F, is indicated by $100v^5_{1\%}$, provided interest is compounded at 1% per period.

Let us consider that the payment of $100v^5_{1\%}$ to my creditor at date A was a loan made to him at 1% per period. Then at date F he would owe me $(100v^5_{1\%})\ (s^5_{1\%})$, and I would owe him \$100. Since the payments of these amounts at date F cancel both debts,

$$100v^5_{1\%} \times s^5_{1\%} = 100$$

and

$$v^5_{1\%} \times s^5_{1\%} = 1.$$

That is, the symbols $s^5_{1\%}$ and $v^5_{1\%}$ are reciprocals. In general, if n and i have the same values in the two symbols, s^n_i and v^n_i are reciprocals, and $v^n_i \times s^n_i = 1$.

Since

$$s_i^n = (1+i)^n,$$

$$v_i^n = \frac{1}{s_i^n} = \frac{1}{(1+i)^n} = (1+i)^{-n}$$

and

$$pv_i^n = p(1+i)^{-n}.$$

Thus

$$v_{2\%}^{20} \times s_{2\%}^{20} = 1.02^{-20} \times 1.02^{20} = 1.$$

But

$$v_{1\%}^{20} \times s_{2\%}^{20} = 1.01^{-20} \times 1.02^{20},$$

which is not 1. The products $v_{1\%}^{20}\ s_{2\%}^{20}$ and $v_{1\%}^{20} s_{1\%}^{8}$ are not 1.

Values of s_i^n and of v_i^n are shown in Tables III and IV for various interest rates and for a number of values of n. The student should familiarize himself with the tables so that he will remember for what values of n and of i the values of the symbols s_i^n and v_i^n are to be found in the tables. We shall see later (page 173) how the value of $(1 + i)^n$ or of $(1 + i)^{-n}$ may be calculated for any values of n and i when tables are not available.

Exercise 58

State in words the meaning of each of the following, and find its value from the tables if possible. ($ signs are understood.)

1. $1000s_{1\%}^{20}$
2. $500v_{\frac{1}{2}\%}^{15}$
3. $100s_{3\%}^{36}$
4. $100v_{4\%}^{56}$
5. $10s_{2\%}^{34}$
6. $100s_{1\%}^{40} \times v_{1\%}^{10}$
7. $1000s_{2\%}^{20} \times v_{2\%}^{50}$
8. $100s_{3\%}^{65} \times v_{3\%}^{35}$
9. $100s_{4\%}^{45} \times v_{4\%}^{60}$
10. $100s_{1\%}^{86} \times v_{1\%}^{53}$

72. Abbreviated multiplication. The values of s_i^n and v_i^n are not given for all consecutive values of n from 1 to 100. But since $s_{2\%}^{48} = 1.02^{48}$ and 1.02^{48} may be written as the product $1.02^{30} \times 1.02^{18}$ or as $1.02^{20} \times 1.02^{28}$, the value of $s_{2\%}^{48}$ may be found by multiplying two numbers that are in

the tables. Similarly, $v_{1\%}^{54} = 1.01^{-54}$ may be written as $v_{1\%}^{30} \times v_{1\%}^{24}$, or as $v_{1\%}^{18} \times v_{1\%}^{36}$. Hence the values of s_i^n and v_i^n, when i is in the tables and n is not, may be found by multiplying numbers obtained from the tables. It is therefore desirable to perform such multiplications expeditiously.

The tables are correct only to a certain number of figures, the last figure being the nearest to the correct one for that place. When numbers are correct only to a certain number of figures, we say that they are correct to so many significant figures. Thus 0.00304 is correct to 3 significant figures, for if the decimal point were disregarded the number would be 304, a number of 3 figures. The number 1.81309 is correct to 6 significant figures, 1.02 is correct to 3 significant figures, and 2.300 is correct to 4 significant figures. If the 2 in 1.02 is exactly right, the number 1.02 is correct to as many significant figures as we please.

It may be proved (Chap. XV) that, if two numbers are correct to different numbers of significant figures, their product cannot be correct to a greater number of significant figures than the less accurate number. Since tables are used extensively, the accuracy of the results of calculations are necessarily limited by the accuracy of the entries in the tables. Much unnecessary labor may be avoided by omitting figures which cannot influence the correctness of the result. Thus to find the product 2.35094 × 1.34078, the decimal points are disregarded and the setup for ordinary multiplication is

23	5094	(1)
13	4078	(2)
18	80752	(3)
164	5658	(4)
9403	76	(5)
70528	2	(6)
235094		(7)
315209	33332	(8)

If the factors are each correct to 6 figures, their product cannot be correct to more than 6 figures and should be written 3.15209. The figures to the left of the vertical line may be obtained more conveniently thus:

235094	(1)
870431	(2)
235094	(3)
70528	(4)
9404	(5)
165	(6)
18	(7)
315209	(8)

(1) One factor is written as ordinarily.

(2) The figures of the other factor are reversed in order, 1 under 4, 3 under 9, and so forth.

(3) The result of multiplying line (1) by 1. Then strike out the 1 and the 4 above it.

(4) Say 3×4 is 12, which requires 1 to carry; $3 \times 9 = 27$, and the carried 1 makes 28, and so on. Strike out 3 and the 9 above it.

(5) Say 4×9 is 36, which is nearer to 40 than to 30, and carry 4.

(6) Nothing is written for the 0 multiplier, but 0 and 5 are stricken out. Then 7×5, or 35, requires that 4 be carried.

(7) 8×3, or 24, requires that 2 be carried.

(8) The result of addition.

In the ordinary multiplication in line (4), the figures to the right of the vertical line, 5658, being omitted, the number to the left of the vertical line is written 165, which is in agreement with line (6) of the abbreviated multiplication. There is a slight discrepancy between line (3) of the ordinary and line (7) of the abbreviated multiplication. Such discrepancies may be avoided by writing the factors

as 2350940 and 1340780 and performing the abbreviated multiplication as before, thus

$$\begin{array}{r}
2350940 \\
\underline{870431} \\
2350940 \\
705282 \\
94038 \\
1645 \\
\underline{188} \\
3152093
\end{array}$$

Since the product cannot be accurate to more than 6 figures, it is 315209.

The position of the decimal point in the product is determined by noting from the given factors that the product is more than 1 and less than 10. The product accordingly is 3.15209.

Exercise 59

1. Use the method of abbreviated multiplication to verify the following:

(a) $s^{20}_{2\%} \times s^{15}_{2\%} = s^{35}_{2\%}$ (c) $v^{20}_{2\%} \times v^{15}_{2\%} = v^{35}_{2\%}$

(b) $s^{20}_{2\%} \times v^{15}_{2\%} = s^{5}_{2\%}$ (d) $v^{20}_{2\%} \times s^{15}_{2\%} = v^{5}_{2\%}$

2. Find the value of each of the following by using the tables:

(a) $s^{46}_{1\%}$ (e) $100 s^{20}_{1\%} \times v^{12}_{1\%}$

(b) $v^{52}_{1\frac{1}{2}\%}$ (f) $100 s^{20}_{1\%} \times v^{12}_{2\%}$

(c) $v^{98}_{2\%}$ (g) $100 s^{30}_{1\%} \times v^{30}_{2\%}$

(d) $s^{150}_{1\%}$ (h) $100 s^{20}_{2\%} \times v^{30}_{2\%}$

3. Martin Pell borrowed $1000 to be repaid at the end of 10 years with compound interest to be calculated at 2% per semiannual period for the first 5 years and at $1\frac{1}{2}\%$ per semiannual period for the next 5 years. Find the amount to be paid at the end of 10 years.

4. Walter Bell owes $1000 due at the end of 5 years with compound interest at 1% per quarter. Find how much he should invest now to be able to pay his debt when due if his investment will accumulate with compound interest at $2\frac{1}{2}\%$ per semiannual period.

73. Abbreviated division. Because 2.35094 × 1.34078 = 3.15209, the result of dividing the product by either factor should give the other factor. The setup for 3.15209 ÷ 2.35094 is:

2350940	3152090
9770431	2350940
	801150
	705282
	95868
	94038
	1830
	1645
	185
	165
	20
	21

Explanation: The quotient cannot be accurate to more than 6 significant figures. As in the multiplication, a zero is added to each of the numbers, and the procedure is: Disregard the decimal points and divide 2350940 into 3152090. Write the quotient 1 under the last figure, 0, of the divisor, multiply the divisor by 1, and subtract. Now strike out the 1 and the 0 above it and say: 801150 ÷ 235094 gives 3, which is placed under the 4. Multiply the divisor by 3 as in the abbreviated method of multiplication, and subtract. Strike out the 3 and the 4 above it and say 95868 ÷ 23509 gives 4, and so on.

The quotient is the number whose figures are written under the divisor in the order in which they were obtained—namely, 1, 3, 4, 0, 7, 7, 9.

The position of the decimal point is determined from the fact that the quotient should be more than 1 but less than 10. That is, the quotient is 1.340779 or, to 6 figures, 1.34078.

Exercise 60

1. Use the tables and the method of abbreviated division to verify the following:

(a) $s^{20}_{2\%} \div s^{40}_{2\%} = v^{20}_{2\%}$
(b) $v^{10}_{1\%} \div v^{40}_{1\%} = s^{30}_{1\%}$
(c) $s^{10}_{3\%} \div v^{10}_{3\%} = s^{20}_{3\%}$
(d) $v^{15}_{4\%} \div s^{8}_{4\%} = v^{23}_{4\%}$

2. Find the value of each of the following by performing the indicated divisions:

(a) $\$2000 \div v^{30}_{2\%}$ (Check by using the relation $\$2000 \div v^{30}_{2\%} = 2000s^{30}_{2\%}$.)
(b) $\$5000 \div s^{25}_{2\%}$
(c) $\$1000 \div v^{50}_{1\%}$.

74. Interpolation. The values of s^n_i and v^n_i are found from the tables for any value of i that is in the table, and for any integral value of n whether it is in the table or not. If, however, i is not in the table but is between two rates that are tabulated, an approximate value of s^n_i or of v^n_i is found by linear interpolation.

Illustrations

1. Find the value of $\$1000s^{20}_{3\frac{1}{3}\%}$. Note that $3\frac{1}{3}\%$ is between 3% and $3\frac{1}{2}\%$, and write:

$x = i$	$y = 1000s^{20}_i$
3%	1806.11
$3\frac{1}{3}\%$	k
$3\frac{1}{2}\%$	1989.79

Since $3\frac{1}{3}\%$ is $\frac{2}{3}$ of the way from 3% to $3\frac{1}{2}\%$, k is $\frac{2}{3}$ of the way from \$1806.11 to \$1989.79 and

$$k = 1806.11 + \tfrac{2}{3}(1989.79 - 1806.11) = 1928.56.$$

The value of k may also be found from

$$k = 1000s^{20}_{3\frac{1}{3}\%} = 1000\,(1.03\tfrac{1}{3})^{20}$$

by using logarithms. But when 4-place logarithm tables are used, only the first 3 figures of the result are reliable.

We shall see later (page 173 and page 177) how the correct value of k, \$1926.63, may be found.

2. Find i from $1250v_i^{15} = 875$.

Division by 1250 gives $v_i^{15} = .7000$.

We find in the table that $v_{2\%}^{15} = .74301$, and $v_{2\frac{1}{2}\%}^{15} = .69047$. Therefore, i is between 2% and $2\frac{1}{2}$%, and we write:

$x = i$	$y = v_i^n$
2%	.74301
h	.70000
$2\frac{1}{2}$%	.69047

Hence $h = 2\% + \dfrac{4301}{5254}$ of $\frac{1}{2}\% = 2.42\%$.

3. Given $100s_{2\%}^{n} = 300$; find n.

Division by 100 gives $s_{2\%}^{n} = 3.0000$.

We find in the table that $s_{2\%}^{50} = 2.69159$ and $s_{2\%}^{60} = 3.28103$.

Therefore, n is between 50 and 60. These values of n are too far apart for a linear interpolation. If, however, we write

$$s_{2\%}^{n} = s_{2\%}^{50+m} = s_{2\%}^{50}\, s_{2\%}^{m} = 3.0000,$$

then

$$s_{2\%}^{m} = \frac{3}{s_{2\%}^{50}} = 3v_{2\%}^{50} = 3(.37153) = 1.11459.$$

Now m is between 5 and 6, since $s_{2\%}^{5} = 1.10408$ and $s_{2\%}^{6} = 1.12616$. Linear interpolation gives $m = 5.48$, and $n = 55.48$.

Exercise 61

Find the values of the unknowns in the following by linear interpolation:

1. $y = s_{2.25\%}^{10}$	**6.** $s_{5\%}^{x} = 2$
2. $s_x^{10} = 1.5$	**7.** $y = v_{3.25\%}^{10}$
3. $y = s_{4\%}^{12.5}$	**8.** $v_x^{10} = 0.68$
4. $800s_x^{25} = 1360$	**9.** $1000s_{1\frac{1}{2}\%}^{n} = 2000$
5. $y = 750v_{1\frac{5}{8}\%}^{30}$	**10.** $y = 750v_{2\frac{1}{8}\%}^{46}$

75. Preview. How an ordinary business transaction gives rise to a problem whose solution requires the use of compound interest symbols may be seen from the consideration of a debt and the provisions for its repayment.

Henry Adams signed a note dated Jan. 5, 1940, acknowledging a debt of $4000 and promising to pay it to the order of Andrew Black. The statement is obviously incomplete since it does not specify when the debt is to be paid, whether interest is to be paid, and, if so, at what rate. The following are some of the possible varieties of payment that may be specified in the note.

1. The debt of $4000 will be paid Jan. 5, 1945, without interest.

2. The debt of $4000 will be paid Jan. 5, 1945, with interest at 4% per annum compounded quarterly.

3. Interest will be paid at the end of every 3 months at 1% per quarterly period, and $4000 will be paid Jan. 5, 1945.

4. The debt of $4000 will be paid without interest by 20 quarterly payments of $200 each, beginning April 5, 1940.

5. Same as 4, except that to each payment of $200 will be added simple interest at 4% per annum from Jan. 5, 1940.

6. Same as 4, except that to each payment of $200 will be added compound interest on the debt then outstanding at 1% per quarterly period.

7. The debt of $4000 will be paid by 20 equal quarterly payments, each payment to be partly for interest on the debt then outstanding and partly toward the reduction of the debt at 1% per quarterly period.

8. The debt of $4000 will be paid by quarterly payments of $200 beginning April 5, 1940, and continuing as long as may be necessary, each payment to be partly for interest at 1% per quarterly period on the debt then out-

standing and partly toward the reduction of the debt. The final balance, less than \$200, will be paid 3 months after the last regular \$200 payment is made.

On July 5, 1940, Charles Gray bought the note from Andrew Black with Black's guaranty that the payments would be made at the date or dates specified. Assuming that Gray will hold the note until the final payment is made, he must know precisely what payments he will receive and the dates of such payments. If Gray bought the note for a sum of money so as to assure him of earning compound interest on his investment at, say, $1\frac{1}{2}\%$ per quarterly period, it is necessary to determine the sum to be paid. If Gray bought the note for a specified sum, say \$3750, it is necessary to determine the rate of interest that he will earn on his investment.

We shall not attempt to solve, at this point, any of the problems that may confront Gray. The problems are presented so that the student may appreciate the necessity for the various topics that he will be required to master.

76. Annuity symbols. Even a superficial examination of the preceding article reveals two methods of paying a debt—namely: (a) by the payment of a single sum, as in 1 and 2, and (b) by making a number of payments.

A number of equal payments made at equal intervals of time and for which the rate of interest remains unchanged is called an *annuity*. Two symbols for the value of an annuity are used. Figure 22 illustrates an annuity of 8 payments of \$10 each at intervals of 3 months. If the rate of interest per quarterly period is 1%, the value of the annuity at date B, immediately after the last payment, is designated by $10s_{\overline{8}|1\%}$ (10 ess angle 8 at 1%), and the value of the same annuity at date A, one period before the first

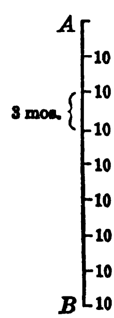


Fig. 22.

payment, is designated by $10a_{\overline{8}|1\%}$ (10 a angle 8 at 1%).

The symbol $s_{\overline{n}|i}$ designates the value, immediately after the last payment, of n payments of 1 each, at equal intervals of time, interest being compounded at rate i for one of the equal time intervals. The symbol $a_{\overline{n}|i}$ designates the value of the same annuity as is indicated by $s_{\overline{n}|i}$ but at a date which is one period before the first payment.

Stated in words, $100s_{\overline{10}|1\%}$ means: If I have an obligation consisting of 10 payments of $100 each at equal intervals of time, and defer every payment until the date when the last payment is due, the amount that I should then pay is $100s_{\overline{10}|1\%}$, provided interest is compounded at 1% for a period that extends from one of the required payments until the next required payment.

The student should state in words the meaning of $100a_{\overline{10}|1\%}$.

Note that, in the symbols s_i^n and v_i^n, n designates the number of periods and i the rate of interest for one of the periods, whereas in $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$, n designates the number of payments. Since there are n periods from the date which indicates one period before the first payment to the date which indicates the last payment, $a_{\overline{n}|i} \times s_i^n = s_{\overline{n}|i}$, and $s_{\overline{n}|i} \times v_i^n = a_{\overline{n}|i}$.

Values of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$, in Tables V and VI, are listed for the same values of n and i as in Tables III and IV.

The four symbols, s_i^n, v_i^n, $s_{\overline{n}|i}$, and $a_{\overline{n}|i}$, will be found sufficient for the solution of all problems involving compound interest or annuities. The value of $s_{\overline{n}|i}$ or of $a_{\overline{n}|i}$ may be found from the tables for any integral value of n whether n is in the table or not, provided i is in the table.

Thus $100s_{\overline{46}|2\%}$ involves an annuity of 46 payments, and $n = 46$ is not in the tables. The 46 payments may, however, be considered as two sets of payments, say 30 payments followed by 16 payments. The value of the

30 payments immediately after the 30th payment is $100s_{\overline{30}|2\%}$, and 16 periods later, at the date of the 46th payment, the value is $100s_{\overline{30}|2\%}s_{2\%}^{16}$. The value of the 16 payments at the date of the last payment is $100s_{\overline{16}|2\%}$. Hence

$$100s_{\overline{46}|2\%} = 100s_{\overline{30}|2\%}\, s_{2\%}^{16} + 100s_{\overline{16}|2\%}.$$

If i is not in the table, use linear interpolation.

Exercise 62

Find the values of the unknowns in the following equations, using Tables III, IV, V, VI, and state each equation in words:

1. $y = 100a_{\overline{15}|1\%}$
2. $y = 100s_{\overline{30}|2\frac{1}{2}\%}$
3. $y = 100s_{\overline{20}|2\%}\, v_{2\%}^{20}$
4. $y = 100a_{\overline{20}|3\%}\, s_{3\%}^{20}$
5. $y = 100a_{\overline{43}|1\%}$
6. $100a_{\overline{30}|x} = 2500$
7. $100a_{\overline{40}|x} = 2050$
8. $100s_{\overline{50}|x} = 8000$
9. $100s_{\overline{24}|x} = 2600$
10. $100s_{\overline{n}|2\%} = 1929.57$

77. Equivalent obligations. A financial obligation states clearly how much money is due and when it is due. A change in the date of payment introduces the element of time, during which interest can be earned. Therefore, when the debtor and the creditor agree to change the date of payment, they must also agree upon the rate of interest to be used in making an equitable adjustment in the amount to be paid. One obligation is compared with another by comparing their values at a given date in accordance with the principle: Two different obligations that, at any arbitrary date, have the same money value, also have equal values at any subsequent or prior date if interest is compounded at the same rate for both obligations. The obligations are then equivalent, and either one may be substituted for the other in any discussion.

Thus if interest is compounded semiannually at 6%

per annum, \$100 due Jan. 1, 1938, and $\$100s^{2}_{3\%}$ or \$106.09, due Jan. 1, 1939, are equivalent. Their values on Jan. 1, 1939, are both $\$100s^{2}_{3\%}$. Their values on Jan. 1, 1942, are $\$100s^{8}_{3\%}$ and $\$106.09s^{6}_{3\%} = \$100s^{2}_{3\%}\,s^{6}_{3\%} = \$100s^{8}_{3\%}$. Their values on Jan. 1, 1936, were $\$100v^{4}_{3\%}$ and $\$106.09v^{6}_{3\%} = \$100s^{2}_{3\%}\,v^{6}_{3\%} = \$100v^{4}_{3\%}$.

Illustration

Carl Thomas owes Frank Till \$1000 due Jan. 1, 1945, with interest at 1% per quarterly period from Jan. 1, 1940. They agree that Thomas shall pay his debt by making 8 quarterly payments of \$100 each, beginning July 1, 1940, and two equal payments on Jan. 1, 1943, and on Jan. 1, 1945. They also agree that the change in the mode of payment shall be based on an interest rate of $1\frac{1}{2}\%$ per quarterly period. Find the amount of each of the equal payments.

Solution

In the diagram, Fig. 23, A indicates Jan. 1, 1940, B is July 1, 1940, C is April 1, 1942, D is Jan. 1, 1943, and E is Jan. 1, 1945. The original obligation, $1000s^{20}_{1\%}$, is shown on the right, due at date E, and the substituted obligation shown on the left indicates 8 payments of \$100 each due at quarterly intervals from B to C, a payment of x at D and another payment of x at E. The two obligations are equivalent and have equal values at any date provided interest is at $1\frac{1}{2}\%$ per quarterly period.

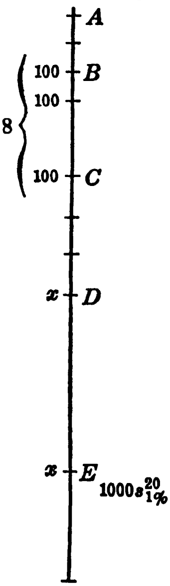


Fig. 23.

The value of the 8 quarterly payments of \$100 each is easily indicated at date C, which is taken as the date at which the comparison is made. Note that the numbers of quarterly periods between the different dates are as follows: A-B, 2; B-C, 7; C-D, 3; D-E, 8. At date C, the value of the new obligation is

$$100s_{\overline{8}|1\frac{1}{2}\%} + xv^{3}_{1\frac{1}{2}\%} + xv^{11}_{1\frac{1}{2}\%}.$$

At the same date C, the value of the original obligation is

$$1000s^{20}_{1\%}\,v^{11}_{1\frac{1}{2}\%}.$$

Hence for equivalence the equation is

$$100s_{\overline{8}|1\frac{1}{2}\%} + xv^3_{1\frac{1}{2}\%} + xv^{11}_{1\frac{1}{2}\%} = 1000s^{20}_{1\%}\, v^{11}_{1\frac{1}{2}\%}$$

and

$$x = \frac{1000s^{20}_{1\%}\, v^{11}_{1\frac{1}{2}\%} - 100s_{\overline{8}|1\frac{1}{2}\%}}{v^3_{1\frac{1}{2}\%} + v^{11}_{1\frac{1}{2}\%}}$$

$$= \frac{1000\,(1.22019)\,(.84893) - 100\,(8.4328)}{(.95632) + (.84893)} = 106.68$$

Each of the equal payments is \$106.68.

The meaning of the equivalence of the new obligation and the old is that, if Till deposits the amounts he receives under the new obligation and his deposits earn interest at $1\frac{1}{2}\%$ per quarter until Jan. 1, 1945, he will have the total necessary to pay the original obligation due at that date.

Proof

The 8 payments of \$100 each accumulate at C to $\$100s_{\overline{8}|1\frac{1}{2}\%} =$ \$843.28 At D this total accumulates to $\$843.28s^3_{1\frac{1}{2}\%} =$ \$881.80, and \$106.68 is added, making the total \$988.48. At E this total accumulates to $\$988.48s^8_{1\frac{1}{2}\%} =$ \$1113.51 when \$106.68 is added, making the final total \$1220.19, which is the amount payable under the terms of the original debt, since $\$1000s^{20}_{1\%} =$ \$1220.19.

Exercise 63

1. P owes Q \$500 due July 1, 1943, and \$1000 due Jan. 1, 1946. The two parties agree that P shall meet his obligation by making two equal payments, one on Jan. 1, 1943, and the other on Jan. 1, 1945. They also agree that money is worth 4% per annum compounded semiannually. How much should each of the equal payments be?

2. Brown holds Dolan's note for \$1000 payable at the end of 10 years without interest, and Dolan holds Brown's note for \$600 due at the end of 10 years with interest at 5% per annum compounded annually and payable when the note matures. The parties agree to settle now on the basis that money is worth 4% per annum compounded quarterly. How much cash is required at the settlement and who pays it?

3. A owes B \$5000 due at the end of 6 years without interest. They agree that A shall pay his debt in three equal payments to be made at the end of 2, 4, and 6 years on the basis that money is worth 4% per annum effective (compounded annually). How much will each payment be?

4. Brown holds Green's note for \$900 payable at the end of 8 years without interest, and Green holds Brown's note for \$800 payable at the end of 5 years without interest. They agree to cancel the notes by a proper cash payment and also agree to calculate interest at i per annum compounded annually. Who shall pay, and how much, if: (a) $i = 4\%$; (b) $i = 5\%$; (c) $i = 6\%$.

5. A owes B the following amounts: \$1000 due at the end of 5 years, and \$3000 due at the end of 8 years, both without interest. They agree to change the mode of payment so that A shall pay his debt by means of 2 equal payments, one at the end of 3 years and the other at the end of 6 years. What should each payment be if interest is to be calculated at: (a) 5% per annum compounded annually; (b) 4% per annum compounded quarterly?

6. Henry Waters agreed to make 16 quarterly payments of \$300 each to John Ray beginning Jan. 1, 1940. Waters and Ray agree that Waters shall settle his debt by making two equal payments, one on Jan. 1, 1942, and the other on Jan. 1, 1945, and that interest shall be at 1% per quarterly period. Find the amount of each payment if:

(a) The agreement was reached before any payments were made by Waters.

(b) The agreement was reached after Waters had made four payments.

7. Jonah Hart holds a note wherein Walter Barley promises to pay \$1000 on Jan. 1, 1945, together with compound interest from Jan. 1, 1940, at $1\frac{1}{2}\%$ per semiannual period. Hart and Barley agree that Barley shall settle his debt:

(a) by a payment of \$200 July 1, 1940, and by making 8 equal semiannual payments beginning Jan. 1, 1941. Interest shall be at $1\frac{1}{2}\%$ per semiannual period. Find how much each payment should be.

(b) by making 8 semiannual payments of \$75 each, beginning Jan. 1, 1941, and an amount x on Jan. 1, 1946. Find x if they agree that interest shall be at $2\frac{1}{2}\%$ per semiannual period.

78. Annuity formulas. In Figure 24, the obligation 1 was assumed at date A, interest payments at rate i per period are to be made at the end of each period, and the debt of 1 is to be paid at date B, n periods after date A.

The two obligations shown on the two sides of the line are equivalent if the interest rate per period is i, and their values are equal at any date. If we compare their values at date B, the value of the obligation on the right is $i\ s_{\overline{n}|i} + 1$, and the value of the obligation on the left is s_i^n.

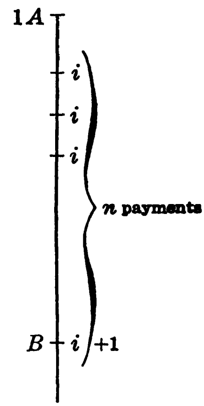


Fig. 24.

Hence

$$i\ s_{\overline{n}|i} + 1 = s_i^n$$

and

$$s_{\overline{n}|i} = \frac{s_i^n - 1}{i} = \frac{(1+i)^n - 1}{i}.$$

If we compare their values at date A, the value of the obligation on the right is $i\ a_{\overline{n}|i} + v_i^n$, and the value of the obligation on the left is 1.

Hence

$$i\ a_{\overline{n}|i} + v_i^n = 1$$

and

$$a_{\overline{n}|i} = \frac{1 - v_i^n}{i} = \frac{1-(1+i)^{-n}}{i}.$$

Furthermore, a payment of 1 at date A may be replaced by an annuity of n payments of $\dfrac{1}{a_{\overline{n}|i}}$ beginning one period after date A, for the value of this annuity at date A is

$$\frac{1}{a_{\overline{n}|i}} \times a_{\overline{n}|i} = 1.$$

A payment of 1 at date B may be replaced by an annuity of n payments of $\dfrac{1}{s_{\overline{n}|i}}$ beginning one period after date A, for the value of this annuity at date B is

$$\frac{1}{s_{\overline{n}|i}} \times s_{\overline{n}|i} = 1.$$

Hence there are two annuities: on one side each payment is $\dfrac{1}{a_{\overline{n}|i}}$, and on the other side each payment is $i + \dfrac{1}{s_{\overline{n}|i}}$.

Therefore,

$$\frac{1}{a_{\overline{n}|i}} = i + \frac{1}{s_{\overline{n}|i}}.$$

Exercise 64

Show by using the formulas for $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ that:

1. $s_{\overline{n}|i} \div a_{\overline{n}|i} = s_i^n$
2. $a_{\overline{n}|i} \div s_{\overline{n}|i} = v_i^n$
3. $1 + i s_{\overline{n}|i} = s_i^n$
4. $1 - i a_{\overline{n}|i} = v_i^n$
5. $\dfrac{1}{s_{\overline{n}|i}} = \dfrac{1}{a_{\overline{n}|i}} - i$
6. $s_{\overline{20}|1\%}\, s_{1\%}^{5} = s_{\overline{25}|1\%} - s_{\overline{5}|1\%}$
7. $a_{\overline{20}|1\%}\, v_{1\%}^{10} = a_{\overline{30}|1\%} - a_{\overline{10}|1\%}$
8. $s_{\overline{20}|1\%} + a_{\overline{20}|1\%} = (s_{1\%}^{20} - v_{1\%}^{20}) \div .01$
9. $s_{\overline{20}|1\%} - a_{\overline{20}|1\%} = .01 s_{\overline{20}|1\%}\, a_{\overline{20}|1\%}$
10. $s_{\overline{48}|1\%} = s_{\overline{40}|1\%}\, s_{1\%}^{8} + s_{\overline{8}|1\%}$
11. $a_{\overline{48}|1\%} = a_{\overline{40}|1\%}\, v_{1\%}^{8} + a_{\overline{8}|1\%}$

12. State in words the meaning of each of the following symbols:

(a) $\$100 s_{2\%}^{10}$
(b) $\$100 v_{1\%}^{10}$
(c) $\$10 s_{\overline{20}|1\%}$
(d) $\$10 a_{\overline{20}|1\%}$
(e) $\$1000 \div s_{\overline{20}|1\%}$
(f) $\$1000 \div a_{\overline{20}|1\%}$
(g) $\$100 a_{\overline{10}|2\%}\, s_{2\%}^{3}$
(h) $\$100 a_{\overline{10}|2\%}\, v_{2\%}^{3}$
(i) $\$100 s_{\overline{20}|3\%}\, v_{3\%}^{5}$
(j) $\$100 s_{\overline{20}|3\%}\, s_{3\%}^{5}$
(k) $\$100 s_{\overline{10}|2\%}\, v_{2\%}^{10}$
(l) $\$100 a_{\overline{10}|2\%}\, s_{2\%}^{10}$

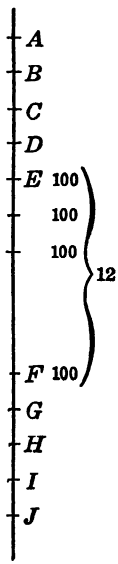


Fig. 25.

79. Value of annuity at various dates. Figure 25 shows an annuity of 12 payments of \$100 each, beginning at date E and ending at date F. If interest is at 2% per period, the values of the annuity are: at D, $100 a_{\overline{12}|2\%}$; and at F, $100 s_{\overline{12}|2\%}$. The value of the annuity at date J, 4 periods after F, is $100 s_{\overline{12}|2\%}\, s_{2\%}^{4}$. The multiplication may be avoided as follows: At dates G, H, I, and J on the diagram, write 100–100. There will then be two annuities ending at J, one of 12 + 4, or 16, payments of 100, and one of 4 pay-

ments of -100, and the value at J is $100s_{\overline{16}|2\%} - 100s_{\overline{4}|2\%} = 100\ (s_{\overline{16}|2\%} - s_{\overline{4}|2\%})$.

The value of the annuity at date A, 3 periods before D, is $100a_{\overline{12}|2\%}\ v^3_{2\%}$. We may again write 100–100 at dates D, C, and B and have two annuities beginning at B, one of $12 + 3$, or 15, payments of 100, and 3 payments of -100. The value at A is $100a_{\overline{15}|2\%} - 100a_{\overline{3}|2\%} = 100\ (a_{\overline{15}|2\%} - a_{\overline{3}|2\%})$.

The student may show by using the formulas for $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ that, algebraically, $100s_{\overline{12}|2\%}s^4_{2\%} = 100\ (s_{\overline{16}|2\%} - s_{\overline{4}|2\%})$ and $100a_{\overline{12}|2\%}\ v^3_{2\%} = 100\ (a_{\overline{15}|2\%} - a_{\overline{3}|2\%})$ are identities. He may also show that no advantage is gained by using this device if it results in an annuity of, say, 43 payments.

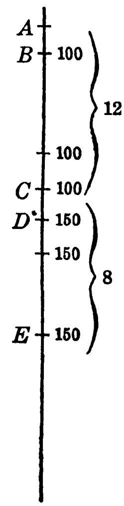


Fig. 26.

80. Unequal payments. The device employed in Article 79 for avoiding multiplications is applicable to such obligations as are indicated in Figure 26, where 12 payments of \$100 each are followed by 8 payments of \$150 each, all the payments being made at equal intervals of time.

If interest is at 1% per period, the value of the 20 payments at date E may be found by replacing each of the \$150 items by \$100 + \$50. There are then 2 annuities ending at E, one of 20 payments of \$100 each beginning at B and one of 8 payments of \$50 each beginning at D. The value of the original 20 payments at date E is $100s_{\overline{20}|1\%} + 50s_{\overline{8}|1\%}$.

The value of the 20 payments at date A is found by forming two annuities, both of which begin at B. By replacing each of the \$100 items by \$150 − \$50, one annuity consists of 20 payments of \$150 each and the other consists of 12 payments of − \$50 each. The value at date A is therefore $150a_{\overline{20}|1\%} - 50a_{\overline{12}|1\%}$.

Exercise 65

1. Deposits of \$500 are made at the end of every 3 months for 5 years. If interest is compounded quarterly at 4% per annum, what will be the amount of the fund at the end of 5 years?

2. If no further deposits are made but the fund accumulates at the same rate of interest, how large will the fund be 3 years after deposits cease?

3. A contract requires a payment of \$200 at the end of every 6 months for 10 years. If interest is compounded semiannually at 6% per annum, what is the value of this contract 6 months before the first payment is made?

4. What is the value of the contract of example 3 three years before the first payment is made?

5. Deposits of \$50 are made at the end of every 3 months for 2 years; during the following 2 years, the quarterly deposits are \$100; and during the next 3 years, the quarterly deposits are \$60 each. If interest is compounded at 1% per quarterly period, find the value of the 28 deposits:

(a) immediately after the 28th deposit is made;
(b) two years after the last deposit is made.

81. Use of the tables. The value of $s^{20}_{1\%} = 1.01^{20}$ may be found by multiplication or by logarithms, and Table III merely enables us to avoid this calculation. But if $s^{20}_i = 1.3$ or $(1 + i)^{20} = 1.3$, the value of i may be found by logarithms, but reference to the table shows at once that i is between 1% and $1\frac{1}{2}$%. The value of $s_{\overline{10}|2\%}$ may be found from s^n_i, Table III, by finding the sum of 10 items—namely, $1 + s^1_{2\%} + s^2_{2\%} + \cdots + s^8_{2\%} + s^9_{2\%}$. The value of $s_{\overline{10}|2\%}$ may also be found by using the formula $s_{\overline{10}|2\%} = \dfrac{s^{10}_{2\%} - 1}{.02}$. For such cases the table for $s_{\overline{n}|i}$ is a convenience. But if $s_{\overline{10}|i} = 11$, the value of i is found from the table to be between 2% and $2\frac{1}{2}$%, whereas without the table for $s_{\overline{n}|i}$ the computation of i would be more difficult.

The tables for s^n_i, v^n_i, $s_{\overline{n}|i}$, $a_{\overline{n}|i}$ are in some cases merely a

convenience and in other cases practically indispensable.

The table for $\frac{1}{a_{\overline{n}|i}}$, Table VII, is convenient for the calculation of $\frac{\$1000}{a_{\overline{20}|2\%}} = \$1000\left(\frac{1}{a_{\overline{20}|2\%}}\right) = \$1000\ (.061157) = \$61.16$. But for $\frac{\$3870.50}{a_{\overline{20}|2\%}}$, no advantage is gained by using the table for $\frac{1}{a_{\overline{n}|i}}$. A table for $\frac{1}{s_{\overline{n}|i}}$ is just as useful as the table for $\frac{1}{a_{\overline{n}|i}}$, but such a table is already available. Since $\frac{1}{a_{\overline{n}|i}} = i + \frac{1}{s_{\overline{n}|i}}$, the value of $\frac{1}{s_{\overline{20}|2\%}}$ is $\frac{1}{a_{\overline{20}|2\%}} - .02 = .061157 - .02 = .041157$.

We shall find presently that the table for $(1 + i)^{1/p}$, Table VIII, for $p = 2, 3, 4, 6, 12$, is necessary in the solution of certain problems. Later (page 174) we shall see how entries in this table as well as in the other tables mentioned may be calculated directly.

Exercise 66

1. Use the appropriate table to find the value of each of the following:

(a) $\frac{\$1000}{s^{20}_{1\%}}$

(b) $\frac{\$1347.50}{v^{20}_{1\%}}$

(c) $\frac{\$1000}{s_{\overline{10}|2\%}}$

(d) $\frac{\$1347.50}{s_{\overline{10}|2\%}}$

(e) $\$1000\ (1.03^{1/3})$

(f) $\frac{\$1000}{1.03^{1/2}}$

(g) $\$1000\ (1.03^{2\ 1/3})$

(h) $\$100\ \frac{1.03^{20} - 1}{1.03^{1/4} - 1}$

2. Find the value of i from each of the following by making a linear interpolation in the appropriate table:

(a) $a_{\overline{20}|i} = 16$

(b) $s_{\overline{20}|i} = 25$

(c) $\frac{1}{a_{\overline{15}|i}} = .08$

(d) $\frac{1}{s_{\overline{20}|i}} = .04$

(e) $(1+i)^{1/3} = 1.005$ (g) $(1+i)^{30} = 1.8$

(f) $(1+i)^{-20} = .6$ (h) $(1+i)^{1/2} = 1.025$

3. Smith pays \$10,000 to an insurance company for a contract whereby he (or his heirs in case of death) will receive 10 equal annual payments, the first of which is to be made 1 year after the date of the contract. If interest is calculated at 3% per annum, compounded annually, how much should each payment be?

4. Henry King, civil service employee, expects to receive regular increases in salary for 15 years. King plans to deposit \$100 at the end of every 3 months for 5 years, then \$200 at the end of every 3 months for 5 years, and then \$150 at the end of every 3 months for 5 years. If the deposits earn interest at 4% per annum compounded quarterly, find how much King will have in the bank (a) at the end of 5 years; (b) at the end of 10 years; (c) at the end of 15 years.

5. King has carried out the plan of example 4. He makes no further deposits for 1 year, but his savings continue to earn interest at the rate stated in example 4. He then decides to withdraw all his savings by drawing equal amounts at the end of every three months for 5 years, beginning 1 year after he made his last deposit at the end of the 15th year. How much can he draw at the end of each quarter?

6. If King had planned to have \$10,000 at the end of 15 years and equal quarterly deposits are to be made during the entire time, find how much each deposit would have to be.

82. Equivalent interest rates. In the symbols $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$, the rate i is for a period that extends from one payment to the next payment. If the given rate is for a period other than the period between payments, the value of i is unknown but may be expressed in terms of the given rate by means of the principle: Two differently stated interest rates are equivalent if at the end of 1 year a dollar will amount to the same total at the 2 differently stated rates.

Thus if 3% per quarterly period and i per semiannual period are equivalent, the relation is $(1 + i)^2 = 1.03^4$, $1 + i = 1.03^2$, and $i = 1.03^2 - 1 = .0609$.

If 3% per semiannual period and i per quarterly period are equivalent, the relation is $(1 + i)^4 = 1.03^2$, $1 + i = 1.03^{1/2}$, $i = 1.03^{1/2} - 1 = .0148892$.

The number of payments per year is usually 1, 2, 4, or 12, and the number of times that interest is compounded per year is also usually 1, 2, 4, or 12. Since any of the modes of making payments may be associated with any one of the modes of compounding interest, the fractional ratios arise $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{12}$. It is for this reason that the table for $(1 + i)^{1/p}$, Table VIII, is made for $p = 2, 3, 4, 6, 12$.

The value of $(1 + i)^{1/p}$ is also necessary for the calculation of $s_{\overline{n}|i}$ when n is not an integer. Thus if Thomas Bank is required to pay interest at 2% per semiannual period, the interest due at the end of 6 months on \$1000 is 2% of \$1000, or \$20. But the interest due at the end of 2 months is not $\frac{1}{3}$ of \$20. The interest rate i for 2 months and the interest rate 2% for 6 months are equivalent and the relation is

$$(1+i)^6 = 1.02^2,$$

or

$$1+i = 1.02^{1/3},$$

and

$$i = 1.02^{1/3} - 1.$$

If in any interest calculation a fractional exponent appears which cannot be reduced to one for which $(1 + i)^{1/p}$ is tabulated, logarithms should be used. For such cases, as well as for cases where the rate of interest is not in the table, the additional table is given for $\log (1 + i)$, Table IX. In this table, the values of i appear from $\frac{1}{8}$ of 1% to 6% at intervals of $\frac{1}{8}$ of 1%.

Illustrations

An annuity consists of 16 quarterly payments of \$100 each and interest is at (a) 3% per semiannual period; (b) $\frac{1}{2}$% per month.

1. To find the value of the annuity at the date of the 16th payment, we write

$$100s_{\overline{16}|i} = 100\,\frac{(1+i)^{16}-1}{i}.$$

(a) 3% per semiannual period and i per quarterly period being equivalent, the relation is

$$(1+i)^4 = 1.03^2,\ 1+i = 1.03^{1/2},\text{ and } i = 1.03^{1/2} - 1.$$

$$100s_{\overline{16}|i} = 100\,\frac{(1+i)^{16}-1}{i} = 100\,\frac{(1.03^{1/2})^{16}-1}{1.03^{1/2}-1} = 100\,\frac{1.03^8-1}{1.03^{1/2}-1}$$

$$= 100\,\frac{1.26677-1}{1.0148892-1} = \frac{26.667}{.0148892} = 1791.8.$$

The value of the annuity is \$1791.80.

(b) $\frac{1}{2}$% per month and i per quarter being equivalent,

$$(1+i)^4 = 1.005^{12},\ 1+i = 1.005^3,\ i = 1.005^3 - 1.$$

$$100s_{\overline{16}|i} = 100\frac{(1+i)^{16}-1}{i} = 100\,\frac{(1.005^3)^{16}-1}{1.005^3-1} = 100\frac{1.005^{48}-1}{1.005^3-1}$$

$$= 100\,\frac{1.005^{50}-1.005^2}{1.005^5-1.005^2} = \frac{.27320}{.01522} = 1795$$

The value of the annuity is \$1795.

2. To find the value of the annuity 3 months before the first payment is made, we write

$$100a_{\overline{16}|i} = 100\,\frac{1-(1+i)^{-16}}{i}.$$

(a) The relation for the equivalent rates is as in (a) of 1, and

$$100a_{\overline{16}|i} = 100\,\frac{1-1.03^{-8}}{1.03^{1/2}-1} = \frac{21.059}{.0148892} = 1414.4.$$

The value of the annuity is \$1414.40.

(b) The relation for the equivalent rates is as in (b) of 1, and

$$100a_{\overline{16}|i} = 100\,\frac{1-1.005^{-48}}{1.005^3-1} = \frac{21.2895}{.01508} = 1412.$$

The value of the annuity is \$1412.

Exercise 67

Under a certain agreement, 20 deposits of $100 each are to be made at intervals of 3 months, and interest is at 6% per annum compounded (a) annually; (b) semiannually; (c) quarterly; (d) monthly.

Find the value of the agreement at the following dates:

1. At the date of the last deposit.

2. Three months before the first deposit is made.

3. Immediately after the 8th deposit (taking into account deposits that are still to be made as well as deposits that have been made).

4. One year before the first deposit is made.

5. Two years after the last deposit.

83. Nominal and effective rates of interest. A rate of interest is stated completely when the rate for a year and the number of conversions per year are given. It is also stated completely when the rate for a period, say 3 months, is given, for it is then understood that conversions are 4 times a year. Whenever the conversion periods are less than a year, the stated annual rate is called the *nominal rate*. The rate for a year that is equivalent to a given rate per period is called the *effective annual rate*. When interest is compounded annually, the nominal and the effective rates are identical. In order that there shall be no misunderstanding, if interest at 4% per annum is to be compounded annually we say that interest is at 4% per annum effective.

Thus if the rate is 1% per quarter and the equivalent effective annual rate is i, then $1 + i = 1.01^4$ and $i = 1.01^4 - 1$.

If the rate is 2% per semiannual period and the equivalent rate is x per quarter, then $(1 + x)^4 = 1.02^2$, $1 + x = 1.02^{1/2}$, and $x = 1.02^{1/2} - 1$. Now the equivalent effective annual rate i may be found either from the quarterly rate or from the semiannual rate. The relation for the quarterly rate is $[1 + (1.02^{1/2} - 1)]^4 = 1 + i$, and

$i = 1.02^2 - 1$, which is also the relation for the semiannual rate.

If the effective annual rate is 4% and the equivalent rate per quarter is x, the relation is $(1 + x)^4 = 1.04$, and $x = 1.04^{1/4} - 1$.

Exercise 68

1. Find the effective annual rate if the given rate is 6% per annum compounded (a) annually; (b) semiannually; (c) quarterly; (d) monthly.

2. Find the nominal annual rate if the effective annual rate is 6% per annum and interest is compounded (a) annually; (b) semiannually; (c) quarterly; (d) monthly.

3. Find the quarterly rate if the given rate is (a) 6% per annum effective; (b) 3% per semiannual period; (c) $\frac{1}{2}$% per month.

4. Find the quarterly rate if interest at 4% per annum is compounded (a) semiannually; (b) annually.

Exercise 69

Review

1. Elmer Bryant owes the following:

(a) $1000 due June 10, 1950, with interest at 5% per annum compounded semiannually from June 10, 1945;

(b) $1000 due June 10, 1952, with interest at 4% per annum compounded quarterly from June 10, 1944.

It is agreed that Bryant shall settle his obligations by making 10 equal semiannual payments, the first of which shall be made on June 10, 1946; and that, for this substitution, interest shall be at 6% per annum compounded semiannually. Find how much each payment should be.

2. Paul Faber holds a contract whereby he is to receive 20 payments of $500 every 3 months, beginning April 15, 1947. Find the value of the contract on January 15, 1946, if interest is calculated at 4% per annum effective.

3. A fund is to be created by making 20 quarterly deposits of $200. If the deposits earn interest at 6% per annum, compounded quarterly, how much will there be in the fund two years after the last deposit?

4. In example 3, the amount of each deposit is unknown, but the final total is to be $5000. How much should each deposit be?

5. Twenty quarterly deposits of $250 were made; the total, immediately after the last deposit was made, was $5600. Find the rate of interest that the fund earned.

6. If $s_{\overline{20}|i} = 15s_i^{20}$, find the value of i.

7. If $2s_i^{10} = 3v_i^5$, find the value of i.

8. 30 semiannual deposits of $100 each produce a fund. Find the value of the fund immediately after the last deposit is made, if interest is calculated:

(a) at 6% per annum effective;
(b) " 4% " " compounded quarterly.

9. Bert Blake borrowed $1000 on January 1, 1940, and was to make 11 monthly payments of $110 beginning February 1, 1940. Find the effective annual rate of interest he paid.

10. Fred Ball agreed to pay $20 on the first of each month beginning March 1, 1940, and ending July 1, 1941, at which time he paid an additional $500. If he paid interest at 12% per annum compounded monthly, how much did he owe February 1, 1940?

11. Find the value of each of the following:

(a) $100s_{3\%}^{42}$	(c) $10s_{\overline{63}|1\%}$
(b) $1000v_{2\%}^{53}$	(d) $10a_{\overline{120}|\frac{1}{2}\%}$

12. Harry Groton makes a contract with an insurance company whereby he (or his heirs) will receive 20 semiannual payments of $500 beginning 5 years after the date of the contract. If interest is at 3% per annum effective, how much should Groton pay?

13. In example 12 the amount of each payment is unknown, but Groton pays $15,000. How much is each semiannual payment?

14. Fred Cotton leased an office for one year at an annual rental of $1200 payable monthly at the beginning of each month. At the end of 6 months his account is clear, and Cotton pays, and his landlord accepts, $550 in full payment of the obligation. Find the rate of interest Cotton earns.

15. Stone & Martin borrowed $5000 on February 1, 1940. They agree to repay the loan by making 20 quarterly payments beginning

May 1, 1940. Find the amount of each payment if interest was at 6% per annum, compounded (a) annually; (b) semiannually; (c) quarterly; (d) every 4 months; (e) monthly.

16. If your bank credits interest at 1% per quarter, find how much you must deposit on February 15, 1940, to permit you to make 20 monthly withdrawals of $100 each beginning May 15, 1941.

17. *A* deposits $50 at the end of every 3 months for 5 years, and then withdraws the entire account by drawing equal amounts at the end of the 7th and 10th years. If the bank allows interest at 1% per quarter, find how much each of the withdrawals should be.

18. *A* owes *B* $1020 due without interest at the end of 5 years, and $2000 due at the end of 6 years with accumulated interest from today at 4% per annum compounded quarterly. *B* owes *A* $3100 due at the end of 4 years without interest. They agree to settle now on the basis that interest is calculated at 5% per annum compounded semiannually. Who pays whom and how much?

19. Gates holds two of Parker's notes, one for $2000 due at the end of 4 years without interest, and one for $1000 due at the end of 6 years with accumulated compound interest at 4% per annum. At the end of 3 years Parker is willing to redeem both notes on the basis that money is worth 6% per annum effective. How much should he pay?

20. To discharge a debt of $15,000 with interest at 4% per annum compounded semiannually, a debtor agrees to pay $5000 at the end of 4 years and equal payments at the end of each 6 months thereafter for 6 years. Find the amount of the semiannual payment.

21. Brown, Dunne and Co. borrowed a sum of money at 6% per annum effective, and will discharge the loan by payments of $500 at the end of each 6 months for 10 years. How much did they borrow?

22. A man makes deposits of $150 at the end of every 3 months for 4 years, followed by quarterly deposits of $100 for the next 3 years. He makes no further deposits for 2 years, but the fund continues to earn interest. He then withdraws his savings in 20 equal quarterly withdrawals, the first of which is paid 2 years after the final deposit of $100. If interest throughout is at 6% per annum compounded quarterly, find the amount of each withdrawal.

23. A debt of $14,000 is to be paid off at the rate of $500 every 3 months, the payments including interest at 1% per quarter. After

10 payments have been made, the debtor is permitted to pay the balance of his debt 3 months later on the basis that money is worth 10% per annum compounded quarterly. How much should he pay?

24. A trust fund of \$60,000 is invested at 5% effective. Payments of \$5000 will be made from the fund at the end of each year as long as possible.

(a) How many full payments of \$5000 each will be made?

(b) How much will be left in the fund after the last full payment of \$5000?

25. The following values for $n = 20$ and $i = 2\%$ are taken from a table:

$$s_i^n = 1.48594740 \qquad a_{\overline{n}|i} = 16.35143334$$

$$v_i^n = .67297133 \qquad \frac{1}{s_{\overline{n}|i}} = .04115672$$

$$s_{\overline{n}|i} = 24.29736980 \qquad \frac{1}{a_{\overline{n}|i}} = .06115672$$

Show that if the value of any one of the above symbols is given, the values of the remaining five symbols may be calculated.

26. Find the value of each of the following symbols by logarithms, using the logarithms in Table IX:

(a) $100s^{20}_{1\frac{5}{8}\%}$

(b) $100v^{20}_{1\frac{5}{8}\%}$

(c) $100s_{\overline{20}|1\frac{5}{8}\%}$

(d) $100a_{\overline{20}|1\frac{5}{8}\%}$

(e) $100s^{20}_{1\frac{5}{8}\%}\ v^{20}_{1\frac{5}{8}\%}$

(f) $100s_{\overline{20}|1\frac{5}{8}\%}\ v^{20}_{1\frac{5}{8}\%}$

(g) $100a_{\overline{20}|1\frac{5}{8}\%}\ s^{20}_{1\frac{5}{8}\%}$

(h) $100a_{\overline{20}|1\frac{5}{8}\%}\ v^{20}_{1\frac{5}{8}\%}$

CHAPTER VII

INVESTMENTS

84. Bonds. Large corporations are financed by the sale of stocks and bonds. The purchaser of a share of stock becomes the owner of a part of the business, whereas the purchaser of a bond becomes a creditor, for a bond is like a promissory note. When bonds are issued, the date of redemption, the dates when interest payments are to be made, and the rate of interest to be paid are specified. Usually a bond has a face value or a *par value* of \$1000, and interest is payable semiannually. Very often a bond is redeemable at a price above par, as at 102, or 102% of \$1000—that is, at \$1020—but for a specified interest rate, say 5%, the semiannual interest payments are $2\frac{1}{2}\%$ of \$1000, or \$25.

A large bond issue is often divided into sets of bonds, each set being redeemable at a different date and perhaps at a different price, but all paying interest at the same rate. Such bonds are called *serial bonds*. An *installment bond* is redeemable in installments, as \$200 at the end of 3 years, \$300 at the end of 5 years, and \$500 at the end of 10 years. Interest on an installment bond is paid on the balance still outstanding and not on the face value.

Interest payments on bonds may be made by checks mailed to the holders. Much inconvenience is avoided, however, by having interest payment notes or *coupons* attached to the bond so that the holder need only cut off the coupons when due and present them for payment. Such a bond, called a *coupon bond*, may be one of a serial

bond issue. The following diagram shows in skeleton form the essential features of a coupon bond.

THE *XYZ* COMPANY will pay to the holder of this bond \$1000.00 July 15, 1945. Interest at 6% per annum will be paid January 15 and July 15. Issued July 15, 1935.	7/15/45 \$30	7/15/43 \$30	7/15/41 \$30	7/15/39 \$30	7/15/37 \$30
	1/15/45 \$30	1/15/43 \$30	1/15/41 \$30	1/15/39 \$30	1/15/37 \$30
	7/15/44 \$30	7/15/42 \$30	7/15/40 \$30	7/15/38 \$30	7/15/36 \$30
	1/15/44 \$30	1/15/42 \$30	1/15/40 \$30	1/15/38 \$30	1/15/36 \$30

85. Value of a bond. Two problems arise in connection with an investment in a bond. The bond may be bought so as to yield a given rate of interest and the price is to be calculated, or the bond may be bought at a given price and the rate of yield is to be calculated. It is assumed that the purchaser will hold the bond to maturity.

If a \$1000 bond is redeemable at 120, or for \$1200, and the semiannual coupons are \$30 each, it is immaterial whether it is stated that the bond pays interest at 3% per period on \$1000 or at $2\frac{1}{2}\%$ on \$1200. A person who buys this bond so as to earn $2\frac{1}{2}\%$ per period on his investment should pay \$1200 for the bond. A person who pays \$1000 for the bond will earn interest on his investment at more than 3% per period.

The following problem is solved by three methods: A \$1000 bond pays interest semiannually at 4% per annum, is redeemable at 105 at the end of 5 years, and is bought to yield $2\frac{1}{2}\%$ per semiannual period. Find the price to be paid for the bond.

Diagram (x) shows the payments that the purchaser will receive. Since his investment is to earn interest at $2\frac{1}{2}\%$ per period, the value to him of the payments in (x) is

$$20a_{\overline{10}|2\frac{1}{2}\%} + 1050v^{10}_{2\frac{1}{2}\%} = 175.04 + 820.26 = 995.30.$$

Diagram (y) shows the payments of a *hypothetical bond* whose redemption value, \$1050, is the same as in the given bond. The interest payments of \$26.25 are each $2\frac{1}{2}\%$ of \$1050, $2\frac{1}{2}\%$ being the rate of interest to be earned by the purchaser. He reasons: A bond such as (y) is worth exactly \$1050, for, if I paid that much for it, my investment would be repaid at date B and I would earn $2\frac{1}{2}\%$ per semiannual period on my investment. But bond (x) is worth less than (y), because each of the 10 interest payments in (x),

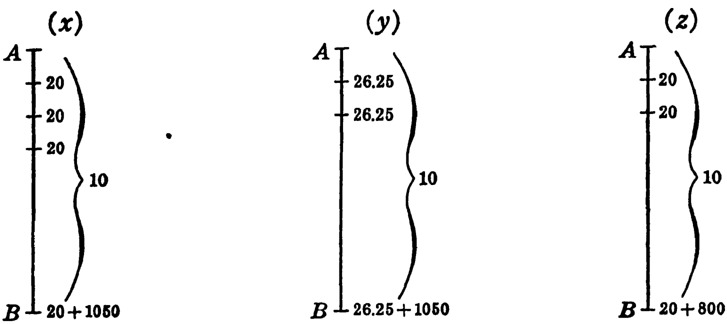


Fig. 27.

\$20, is less than each one in (y), \$26.25, by \$6.25. The value at date A of 10 payments of \$6.25 each is $6.25a_{\overline{10}|2\frac{1}{2}\%}$. Therefore the value of bond (x) at date A is

$$1050 - 6.25a_{\overline{10}|2\frac{1}{2}\%} = 1050 - 54.70 = 995.30.$$

Diagram (z) shows the payments of a *hypothetical bond* whose interest payments are the same as those of the given bond. The redemption value, \$800, is found from $\frac{\$20}{.025}$, so that $2\frac{1}{2}\%$ of \$800 = \$20. The reasoning is: Bond (z) is worth exactly \$800, for, if I paid that much for it, my investment would be repaid at date B and I would earn $2\frac{1}{2}\%$ per semiannual period on my investment. But bond (x) is worth more than (z), because its redemption value, \$1050,

exceeds that of (z), \$800, by \$250. The value at date A of \$250 payable at date B is $250v^{10}_{2\frac{1}{2}\%}$, and the value of bond (x) at date A is

$$800 + 250v^{10}_{2\frac{1}{2}\%} = 800 + 195.30 = 995.30.$$

The three formulas for the value of the bond, namely,

$$20a_{\overline{10}|2\frac{1}{2}\%} + 1050v^{10}_{2\frac{1}{2}\%},\ 1050 - 6.25a_{\overline{10}|2\frac{1}{2}\%},\ 800 + 250v^{10}_{2\frac{1}{2}\%},$$

are algebraically identical. The first formula requires the use of two tables. Only one table is needed for the second or for the third formula.

The following tabulation shows how the investment of \$995.30 increases periodically, because the interest earned at $2\frac{1}{2}\%$ is more than the coupon, until the investment reaches the redemption value.

Semiannual Period	*Value at Beginning of Period*	*Interest at $2\frac{1}{2}\%$ for the Period*	*Coupon*	*Additional Investment*
1	995.300	24.883	20.000	4.883
2	1000.183	25.005	20.000	5.005
3	1005.188	25.130	20.000	5.130
4	1010.318	25.258	20.000	5.258
5	1015.576	25.389	20.000	5.389
6	1020.965	25.524	20.000	5.524
7	1026.489	25.662	20.000	5.662
8	1032.151	25.804	20.000	5.804
9	1037.955	25.949	20.000	5.949
10	1043.904	26.098	20.000	6.098
11	1050.002			

A bond bought at a price below the face value is said to be bought at a *discount*. A bond bought at a price above the face value is said to be bought at a *premium*. In such cases each coupon may be more than the interest to which the purchaser is entitled, the excess representing so much of his investment repaid at the coupon date.

Exercise 70

Find the value of each of the following $1000 bonds paying interest semiannually by each of the three methods.

	Annual Interest Rate	*Years to Mature*	*Redemption Price*	*Rate of Yield Per 6 Mo. Period*
1.	5%	12	$1000	3%
2.	5%	12	1000	2%
3.	$4\frac{1}{2}$%	15	1010	2%
4.	$4\frac{1}{2}$%	15	1010	$2\frac{1}{2}$%
5.	$3\frac{3}{4}$%	10	1025	2%
6.	$3\frac{3}{4}$%	10	1025	$1\frac{1}{2}$%
7.	$5\frac{1}{2}$%	8	1000	$3\frac{1}{2}$%
8.	$5\frac{1}{2}$%	8	1000	$2\frac{1}{2}$%
9.	$4\frac{1}{4}$%	12	1030	2%
10.	$4\frac{1}{4}$%	12	1030	3%

11. Construct tables like that on page 149 for the bonds of examples 1 and 2.

86. Rate of yield not in tables. If the bond discussed in Art. 85, page 148, is bought to yield $2\frac{2}{3}\%$ per semiannual period, the values of the bond found by linear interpolation from the three formulas are

$$20a_{\overline{10}|2\frac{2}{3}\%} + 1050v^{10}_{2\frac{2}{3}\%} = 980.83;$$
$$1050 - 8a_{\overline{10}|2\frac{2}{3}\%} = 980.58;$$
$$750 + 300v^{10}_{2\frac{2}{3}\%} = 980.65.$$

The three results are different because the values of $a_{\overline{10}|2\frac{2}{3}\%}$ and $v^{10}_{2\frac{2}{3}\%}$ found by linear interpolation are too large. The second formula gives a value that is too small, whereas the first and the third formulas give values that are too large. The error in using the second formula is less than that in using the third, and either of these results is more nearly correct than the result of the first formula because in the first formula two interpolations are necessary.

Note that in our tables interest rates are at intervals of

$\frac{1}{2}\%$. Tables are obtainable in which the interest rates are at smaller intervals, and linear interpolation in such tables gives results that are more nearly correct.

If the bond is bought to yield 5% per annum effective, it is first necessary to find the rate per semiannual period, r, equivalent to 5% per annum effective, from $(1 + r)^2 = 1.05$, or $r = .0246951$. Now two methods of solution are possible.

(a) Use the formula

$$20a_{\overline{10}|r} + 1050v_r^{10} = 20\frac{1-(1+r)^{-10}}{r} + 1050\,(1+r)^{-10} = 20\frac{1-1.05^{-5}}{1.05^{1/2}-1} + 1050 \times 1.05^{-5} = 998.02.$$

(b) Make an equivalent bond in which the interest payments are made annually. Since the investor is to earn 2.46951% per semiannual period, let the payment of the first semiannual coupon each year be deferred 6 months, earning 49.39 cents interest during this period.

The given bond is therefore equivalent to a bond having 5 annual coupons of \$40.4939 and redeemable at the end of 5 years for \$1050. The value is

$$1050 - 12.0061a_{\overline{5}|5\%} = 998.02.$$

Exercise 71

Find the values of the following \$1000 bonds that pay interest semiannually:

	Annual Interest Rate	*Years to Mature*	*Redemption Price*	*Rate of Yield*
1.	5%	12	\$1000	$3\frac{1}{3}\%$ per half year
2.	5%	12	1000	6% per annum effective
3.	$4\frac{1}{2}\%$	15	1010	$2\frac{1}{4}\%$ per half year
4.	$4\frac{1}{2}\%$	15	1010	4% per annum effective
5.	$3\frac{3}{4}\%$	10	1025	$1\frac{3}{4}\%$ per half year
6.	$3\frac{3}{4}\%$	10	1025	$3\frac{1}{2}\%$ per annum effective
7.	$5\frac{1}{2}\%$	8	1000	2.3% per half year
8.	$5\frac{1}{2}\%$	8	1000	4% per annum effective

9.	$5\frac{1}{2}\%$	12	1030	4% per annum effective
10.	$5\frac{1}{2}\%$	12	1030	6% per annum effective

87. Bonds bought at a given price. Bonds are usually bought at a given price, and the yield rate is required. The solution of the problem is very simple if bond tables are available.

If, however, bond tables are not available, the procedure is as shown in the following

Illustration

A \$1000 bond pays interest semiannually at 4% per annum, is redeemable at 105 at the end of 5 years, and is bought at $92\frac{1}{2}$, or for \$925. Find the rate of yield.

During the life of the bond, the purchaser will collect $10 \times \$20 + \$1050$, or \$1250. Since he pays \$925, his profit for 10 periods is \$325, and he will have an *average income* of \$32.50 per period.

His initial investment is \$925, but finally at the end of 5 years his investment will be worth the redemption price of the bond, or \$1050. His *average investment* is

$$\tfrac{1}{2}(925 + 1050) = 987.50.$$

The *average rate of yield* per period is

$$\frac{\text{average income per period}}{\text{average investment}} = \frac{32.50}{987.50} = 3.29\%.$$

This result is only an approximation, and must be corrected. The rate near to the average rate in our tables is 3%. At this rate the value of the bond is

$$1050 - 11.50a_{\overline{10}|3\%} = 951.90.$$

Since the cost—namely, \$925—was less than \$951.90, the rate of yield was higher than 3%.

At $3\frac{1}{2}\%$, the next higher rate in the table, the value of the bond is

$$1050 - 16.75a_{\overline{10}|3\frac{1}{2}\%} = 910.70.$$

We are now certain that the yield rate per period is between 3% and $3\frac{1}{2}\%$ and find by linear interpolation in the table:

3%	951.90
x	925.00
$3\frac{1}{2}\%$	910.70

$$x = 3\% + \frac{26.90}{41.20} \times \tfrac{1}{2}\% = 3.326\%.$$

Exercise 72

Find the rate of interest earned per semiannual period on each of the following $1000 bonds that pay interest semiannually.

	Coupon	*Number of Coupons*	*Redemption Price*	*Purchase Price*
1.	$25	20	$1000	$1000
2.	25	20	1000	900
3.	25	20	1000	1100
4.	20	30	1050	1050
5.	20	30	1050	900
6.	20	30	1050	1100
7.	22.50	24	1025	1025
8.	22.50	24	1025	900
9.	22.50	24	1025	1100
10.	18.75	32	1100	1000
11.	18.75	32	1100	900

88. Bond bought between interest dates. The problems considered thus far have been with bonds bought at an interest date. Most purchases, however, are made between interest dates, and adjustments in price must be considered. *The following procedure is the common practice.*

The time between two dates, say from April 1 to July 16, is found by writing these dates as 4–1 and 7–16, and then subtracting one from the other, giving 3 months and 15 days (see page 13). Then 30 days are counted to the month, giving 105 days, which is $\frac{105}{180} = \frac{7}{12}$ of an interest period, or $\frac{7}{24}$ of a year.

Now suppose that in the bond discussed in Art. 85 the coupon dates are April 1 and October 1, that date A in the diagrams, page 148, is April 1, 1940, and that the bond was bought July 16, 1940, to yield $2\frac{1}{2}\%$ per semiannual period.

The value of the bond on April 1, 1940, is \$995.30. Since the bond was bought July 16, 1940, the purchaser is charged *simple interest* on \$995.30 at 5% per annum for $\frac{7}{24}$ of a year, or $\$995.30 \times .05 \times \frac{7}{24} = \$14.51$. He pays $\$995.30 + 14.51 = \$1009.81$. The extra charge of \$14.51 is called *accrued interest* and is the result of two items: (a) the increase in the book value of the bond during the $3\frac{1}{2}$ months, and (b) the necessity of apportioning the coupon of \$20 payable October 1, of which $\frac{7}{12}$ is due to the preceding holder.

If the bond is quoted at a price, say 92, the purchaser pays \$920 and $\frac{7}{12}$ of the \$20 coupon, or $\$920 + 11.67 = \$931.67$.

The effect of the adjustments in price made by these methods is that the purchaser pays a few cents more for a \$1000 bond than he would pay if interest were compounded.

Exercise 73

Find the price to be paid for each of the following \$1000 bonds redeemable at par and paying interest semiannually.

	Coupon Dates	*Coupons*	*Maturity Date*	*Date Purchased*	*Purchased at*
1.	1/15,7/15	\$25	7/15/50	3/10/40	4% yield
2.	2/15,8/15	30	8/15/46	4/20/40	5% yield
3.	3/10,9/10	20	9/10/52	11/15/39	6% yield
4.	4/1,10/1	22.50	4/1/51	1/10/40	$87\frac{1}{2}$
5.	5/15,11/15	17.50	11/15/48	3/6/40	91

89. Bond tables. Bonds are bought and sold daily and are usually quoted at so many per cent of par value. Thus a bond quoted at $91\frac{3}{4}$ means a price of \$917.50 for a \$1000 bond. Dealers in bonds make their calculations by using bond tables. There are many excellent bond tables to be had in which the value of a one-million-dollar bond is given to the nearest cent for yield rates that vary by small

fractions and for bonds whose life runs to 50 years or more.

Table X, a small part of such a bond table shows the value of a 10-year, \$1000 bond redeemable at par. The coupon rate and the yield rate are given for a year so that for a semiannual period the rates are $\frac{1}{2}$ of the rates tabulated. In this table we may find at once the price to be paid for a 10-year bond if the coupon rate and the yield rate are given. We may also find the yield rate when the purchase price and the coupon rate are given. In either case it may be necessary to make a linear interpolation.

Illustrations

1. The price to be paid for a \$1000 10-year bond paying interest semiannually at 5% per annum and yielding 3.6% per annum compounded semiannually is \$1116.70.

2. The yield rate of a \$1000 10-year bond paying interest semiannually at 5% per annum and bought for \$925.61 is 6% per annum compounded semiannually.

3. If the bond of Illustration 2 is bought for \$950, the yield rate is between 5.6% and 5.8%. Linear interpolation gives 5.66%.

The bond table is constructed by using the formula

$$P = 1000\, v_i^n + c a_{\overline{n}|i}$$

where n = the number of semiannual coupons, c = the amount of each coupon, i = the yield rate per semiannual period, and P = the price paid. Therefore for given values of n and i, the values of P and c are related linearly and a linear interpolation for a coupon rate not in the table gives a correct value of P. Thus the values for a 4% yield rate increase by \$40.88 for each increase of $\frac{1}{2}\%$ in the coupon rate.

Furthermore the values of v_i^{20} and $a_{\overline{20}|i}$ may be found from the bond table for any yield rate, say 5.2%, by taking any two coupon rates, say 4% and 6%. Thus if $P_{4\%}$ is the price of a 4% bond,

$$P_{4\%} = 1000 v_{2.6\%}^{20} + 20 a_{\overline{20}|2.6\%} = 907.34,$$

and

$$P_{6\%} = 1000 v_{2.6\%}^{20} + 30 a_{\overline{20}|2.6\%} = 1061.77$$

These simultaneous equations in $v^{20}_{2.6\%}$ and $a_{\overline{20}|2.6\%}$ are easily solved, giving $a_{\overline{20}|2.6\%} = 15.443$ and $v^{20}_{2.6\%} = .59848$.

The price to be paid for a *premium bond,* a bond redeemable above par, may now be found. Thus the price of a 10-year, 4%, \$1000 bond redeemable at 106 and bought to yield 5.2% is more than that of a bond redeemable at par—namely \$907.34—by the present value of the premium, \$60, which will be received at the end of 10 years. The value of this premium is $60v^{20}_{2.6\%} = 35.91$ and the price of the premium bond is $907.34 + 35.91 = 943.25$.

A premium bond may be changed to a corresponding bond redeemable at par. Thus a 10-year, 4%, \$1000 bond redeemable at 106% and bought for \$850 is changed to a 10-year, $\frac{4}{1.06}\%$, \$1000 bond redeemable at $\frac{106}{1.06}\%$ and bought for $\frac{\$850}{1.06}$. That is, the premium bond is changed to a 10-year, 3.7736%, \$1000 bond redeemable at par and bought for \$801.89. The annual yield rate is the same for both bonds. Thus the price to be paid for the premium bond discussed in the preceding paragraph may also be found as follows: The value of a 10-year, 3.7736%, \$1000 bond redeemable at par and bought to yield 5.2 per annum is

$$868.74 + \tfrac{2736}{5000} \times 38.60 = 889.86$$

The value of the premium bond is $889.86 \times 1.06 = 943.25$.

If a premium bond is bought at a given price, and the rate of yield is required, we first change the premium bond to the corresponding bond redeemable at par bought at a corresponding price. We then find the approximate yield rate from

$$\text{approximate yield rate} = \frac{\text{average income}}{\text{average investment}},$$

then we find the correct yield rate by interpolation.

Thus a 10-year, 4%, \$1000 bond redeemable at 106 and

bought for \$850 is changed, as before, to a 10-year, 3.7736% \$1000 bond redeemable at par and bought for \$801.89. The approximate yield rate is 6.4%. Since the coupon rate, 3.7736%, is between $3\frac{1}{2}\%$ and 4%, the value at 6.4% yield is

$$788.21 + \tfrac{2736}{5000} \times 36.52 = 808.19\cdot$$

at 6.6% yield the value is

$$775.67 + \tfrac{2736}{5000} \times 36.18 = 795.47.$$

A price of \$801.89 gives, by interpolation, a yield rate of 6.50%.

Exercise 74

Make the calculations required in the following examples relating to a \$1000 bond redeemable at the end of 10 years by using the bond table:

1. Find the price to be paid if the bond is redeemable at par, the coupon rate is 5%, and the yield rate is (a) 4.5%; (b) 6.75%; (c) 3.7%.

2. Find the yield rate if the coupon rate is 4%, the bond is redeemable at par and is bought at (a) 90%; (b) 104%; (c) 108%.

3. Calculate the values of $a_{\overline{20}|i}$ and v_i^{20} if (a) $i = 3\frac{1}{4}\%$; (b) $i = 2\frac{3}{4}\%$; (c) $i = 2\frac{3}{8}\%$.

4. Solve example 1 if the bond is redeemable at (a) 105%; (b) 110%.

5. Solve example 2 if the bond is redeemable at (a) 105%; (b) 110%.

6. The bond of example 1, with 20 coupons attached, is bought 40 days after the date of issue, or of the last preceding coupon. Find the price paid if the yield rate is (a) 4.5%; (b) 6.75%; (c) 3.7%.

90. Serial bonds. Brock and Co. issued 500 bonds, each having a face value of \$1000, and paying interest semiannually at 4% per annum, so that the coupons were for \$20 each. The bond issue provided for the redemption at par of 25 bonds at each interval of 6 months, beginning 6 months after the date of issue, so that all the bonds would be redeemed at the end of 10 years.

Fuller and Ball bought the entire \$500,000 issue at a price that would enable them to earn interest at $2\frac{1}{2}\%$ per semiannual period, and intended to sell the bonds in large or

in small lots to long-term or to short-term investors. The issue consists of 20 different varieties of bonds, and Fuller and Ball wish to know how much they should pay to Brock and Co.

Diagram (x) shows the payments that Brock and Co. will make, and (y) shows a *hypothetical issue* in which the interest payments are exactly as in (x), but in which each principal payment is \$20,000 instead of \$25,000.

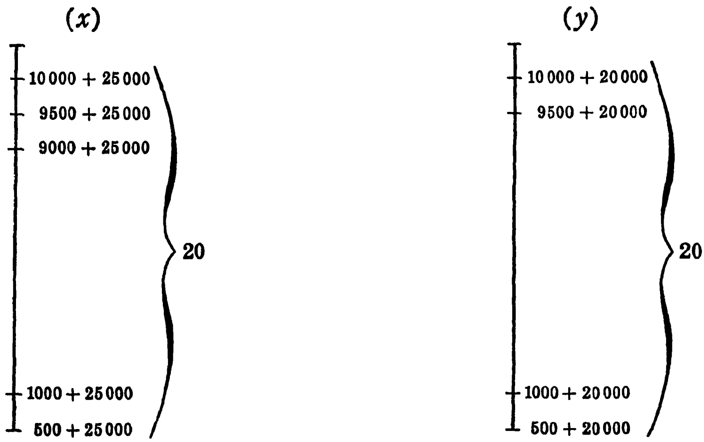


Fig. 28.

Since Fuller and Ball wish to earn interest at $2\frac{1}{2}\%$ per period, a reduction of \$500 in interest corresponds to a reduction of $\frac{\$500}{.025}$, or \$20,000, in principal, for $2\frac{1}{2}\%$ of \$20,000 = \$500.

They argue: We would be willing to pay \$400,000 for an issue such as (y) because then our investment would be returned by 20 payments of \$20,000 each, and we would receive interest on our outstanding investment at the rate we desire, $2\frac{1}{2}\%$ per semiannual period. That is, (y) is worth to us exactly \$400,000. But (x) is worth more than (y) by the value of a set of 20 semiannual payments of

\$5000, which at date A is \$5000 $a_{\overline{20}|2\frac{1}{2}\%}$. We will therefore pay

$$\$400,000 + 5000\, a_{\overline{20}|2\frac{1}{2}\%} = \$477,946.$$

Thus instead of having to calculate the value at date A of 20 bonds having different maturity dates, the analysis shows that *only one* simple calculation is necessary. (The student may show that the formula also gives the value of 400 bonds having 20 coupons of \$37.50.)

Exercise 75

1. Find the price to be paid by Fuller and Ball if the interest per semiannual period is to be (a) 3%; (b) $3\frac{1}{2}\%$; (c) $1\frac{3}{4}\%$.

2. The provision for redemption is that 25 bonds will be redeemed at semiannual intervals beginning 5 years after the date of issue. Find the value of the issue if interest earned on the investment per semiannual period shall be (a) $1\frac{1}{2}\%$; (b) $2\frac{1}{2}\%$; (c) 3%; (d) $3\frac{1}{2}\%$.

3. The provision for redemption is that 25 bonds are redeemed at annual intervals beginning 5 years after the date of issue. Find the value of the issue if interest on the investment per semiannual period shall be (a) $1\frac{1}{2}\%$; (b) $2\frac{1}{2}\%$; (c) 3%; (d) $3\frac{1}{2}\%$.

4. A million-dollar issue of \$1000 bonds provides for quarterly interest payments at $\frac{3}{4}\%$ per quarter and for the redemption at par of 10 bonds at the end of each quarter beginning 3 months after the date of issue. Find the price if the entire issue is bought by a life insurance company at a price which will enable it to earn interest on its investment (a) at 1% per quarterly period; (b) at 4% per annum effective; (c) at $\frac{1}{2}\%$ per quarterly period; (d) at $3\frac{1}{2}\%$ per annum effective.

5. A \$100,000 issue of \$1000 bonds provides for semiannual interest payments at $2\frac{1}{2}\%$ per semiannual period, and for the redemption of 10 bonds at par at the end of every two-year period. Find how much should be paid for the entire issue in order to earn interest on the investment (a) at 3% per semiannual period; (b) at 2% per semiannual period; (c) at 5% per annum effective.

91. Serial bond issue bought at a price. The bond issue discussed on page 158 was bought by Fuller and Ball

for \$460,000. To find the rate of yield, an approximate rate is first obtained from

$$\text{average rate} = \frac{\text{average income}}{\text{average investment}}.$$

To find the average income, it is necessary to add the 20 interest items: 10,000, 9500, 9000, . . . , 1500, 1000, 500. A set of numbers that increase or decrease steadily by the same amount is an arithmetic progression whose sum is easily found by pairing the numbers, the first with the last, the second with the one before the last, and so on. The sum of each pair in this example is 10,500, and there are 10 pairs. Hence the sum of these 20 items is 105,000. The total income is \$105,000 + 20 × 25,000 − 460,000, or \$145,000 earned in 10 years. Hence the average income per year is \$14,500.

The initial investment is \$460,000, and the final investment is \$25,000. Hence the average investment is $\frac{1}{2}$(\$460,000 + 25,000) = \$242,500. The average rate is $\frac{14,500}{242,500}$, or 5.98% per year.

The value of the bond issue at 3% per half year is

$$\frac{1,000,000}{3} + \frac{25,000}{3}\, a_{\overline{20}|3\%} = 457,312.50.$$

Since more than 457,312.50 was paid, the rate of yield is less than 3% per half year. At $2\frac{1}{2}$%, page 159, the value is 477,946. Now find the annual rate x by linear interpolation in the table:

5%	477946
x	460000
6%	457313

and

$$x = 5\% + \frac{17946}{20633} \text{ of } 1\% = 5.87\%.$$

Hence the rate of yield is 5.87% per annum compounded semiannually.

(The student may show that the formula also gives the value of $333\frac{1}{3}$ bonds having 20 coupons of \$55.)

Exercise 76

1. A serial bond issue of \$100,000 provides for the redemption of ten \$1,000 bonds at the end of each year beginning 3 years after the date of issue, and for semiannual payments of interest at $3\frac{1}{2}\%$ per annum. Find the rate of yield if the issue is bought for (a) \$95,000; (b) \$90,000; (c) \$86,000.

2. The bond issue of example 1 pays interest semiannually at $4\frac{1}{2}\%$ per annum. Solve the problem if all other specifications remain unchanged.

3. Solve example 1 if the only change is that the redemption begins one year after the date of issue.

92. Unequal payments. An annuity, a succession of equal payments made at equal intervals of time, is easily evaluated at any date and at any interest rate. If the payments are not equal, it is generally necessary to evaluate each item separately. But if the successive items decrease or increase by the same amount, or are in arithmetic progression, a simple formula for the value of all the items is easily obtained by the method of analysis employed in the valuation of serial bonds.

Illustrations

1. Twenty periodic payments (in dollars) are, successively,

$$60, 57, 54, \ldots, 12, 9, 6, 3.$$

Find the value, V, of these payments at date A, one period before the first payment, if interest is at 2% per period. Diagram (x), Fig. 29, represents the given payments, having an unknown value V at date A. (y) is constructed by considering that the payments in (x) represent interest payments. Then each decrease of \$3 in interest corresponds to a decrease of \$150 in principal, since 2% of \$150 = \$3.

(y) therefore represents a debt of \$3000 which is to be paid in 20 periodic installments of \$150 each and which also requires periodic interest payments on the outstanding debt at 2% per period.

The value of (y) at date A, at 2% per period, is \$3000.
But it is also $V + 150a_{\overline{20}|2\%}$. Hence

$$V + 150a_{\overline{20}|2\%} = 3000, \text{ and } V = 3000 - 150a_{\overline{20}|2\%}.$$

2. The order of the 20 periodic payments is reversed as in (z) and the value, V', of these payments is required at date A.

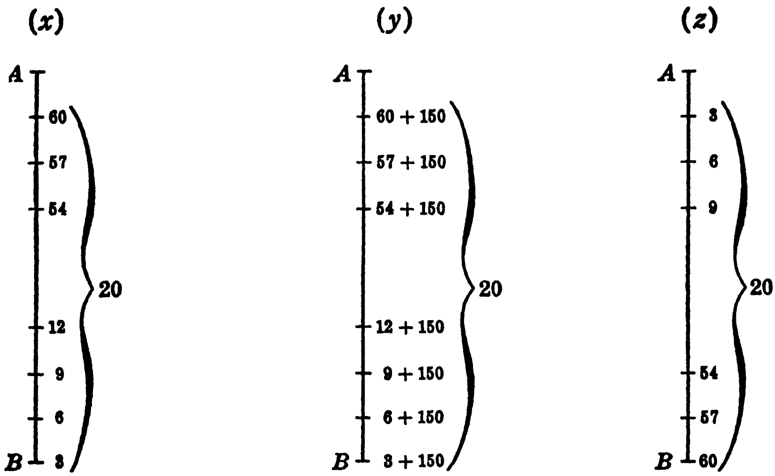


Fig. 29.

From (x) and (z) it is clear that the sum of each pair of corresponding payments is 63, or

$$V + V' = 63a_{\overline{20}|2\%}.$$

Hence $$V' = 63a_{\overline{20}|2\%} - (3000 - 150a_{\overline{20}|2\%})$$

or $$V' = 213a_{\overline{20}|2\%} - 3000.$$

93. Real estate mortgages. A real estate mortgage differs from a bond in that interest is usually payable quarterly and that it is never redeemable above par. Amortized mortgages are very much like serial bonds, the principal or part of it being paid off gradually. But a mortgage can be so drawn as to be one or another of the types of problems that have been considered when a debt

is to be paid off by some particular set of payments. The methods of calculating the value of a mortgage when it is bought so as to yield a given rate of interest or of calculating the rate when it is bought at a given price do not require further discussion since such problems have already been discussed.

Exercise 77

1. A mortgage of $12,000 is payable at the end of 8 years, and interest is payable quarterly at 4% per annum. Find the price to be paid to yield (a) $1\frac{1}{2}$% per quarter; (b) 6% per annum effective.

2. Find the rate of yield if the mortgage of example 1 is bought for (a) $11,500; (b) $8,500.

3. A mortgage of $12,000 is payable in 40 equal quarterly installments, beginning 3 months after it was issued, each installment being partly for interest at 1% per quarter and partly for principal. Find the amount due at the end of each quarter.

4. Use the answer to example 3 and find the value of the mortgage to yield (a) $1\frac{1}{2}$% per quarter; (b) 6% per annum effective.

5. Use the answer to example 3 and find the yield rate if the mortgage is bought for (a) $11,500; (b) $8,500.

6. A mortgage of $12,000 is payable in quarterly installments of $500, beginning 3 months after the date of issue, each installment being partly for interest at 1% per quarter and partly for principal.

(a) Find the number of payments of $500 that will be required.

(b) Find the balance still due after the last $500 payment has been made.

(c) Find how much should be added to each $500 payment so that there will be no balance.

7. Use the answers to (a) and (b) of example 6 and find the price to be paid for the mortgage to yield (a) $1\frac{1}{2}$% per quarter; (b) 6% per annum effective.

8. Use the answers to (a) and (b) of example 6 and find the yield rate if the mortgage is bought for (a) $11,500; (b) $8,500.

9. A mortgage for $12,000 is payable in quarterly installments of $400 on account of principal. Interest at 1% per quarter is also pay-

able at the time installments of principal are paid. How much is the total amount that the mortgage holder will receive?

10. In example 9, if the mortgage holder deposits each payment when he receives it, find how much he will have when he receives the final payment if his deposits earn interest at (a) 1% per quarter; (b) $1\frac{1}{2}$% per quarter.

11. Find how much should be paid for the mortgage of example 9 for the investor to earn interest at (a) $1\frac{1}{2}$% per quarter; (b) 2% per quarter.

12. Find the yield rate to an investor who buys the mortgage of example 9 for (a) $11,500; (b) $8,500.

13. A mortgage of $12,000 requires quarterly interest payments at 1% per quarter and 6 annual payments of $2,000 on account of principal beginning one year after the date of issue. Find the value of the mortgage if the investor is to earn 2% per quarter on his investment.

14. Find the rate of yield to the purchaser of the mortgage of example 13 if he pays $10,000 for it.

15. A mortgage of $12,000 is to be paid off as follows: 8 quarterly payments of $500 beginning 3 months after the date of issue, each payment including interest at 1% per quarter; then 8 quarterly payments of $600 and also interest at 1% per quarter; the balance is due 2 years after the last $600 payment, with compound interest at 1% per quarter. Find how much the final payment should be.

CHAPTER VIII

METHODS OF CALCULATION

94. Progressions. A sequence of numbers such as 2, 5, 8, 11, and so forth is called an *arithmetic progression*, or an *AP*. The differences $5 - 2$, $8 - 5$, $11 - 8$ are the same number, 3, called the *common difference*. In the *AP* 5, 3, 1, -1, and so on, the common difference is -2. The general form of an *AP* is

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,$$

where d is + for an increasing *AP* and − for a decreasing *AP*.

A sequence such as 2, 6, 18, 54, and so on is called a *geometric progression*, or a *GP*. The ratios $\frac{6}{2}$, $\frac{18}{6}$, $\frac{54}{18}$ are the same number, 3, called the *common ratio*. In the *GP* 6, 3, $\frac{3}{2}$, $\frac{3}{4}$, the common ratio is $\frac{1}{2}$. The general form of a *GP* is

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,$$

where the numerical value of r is more than 1 for an increasing *GP* and less than 1 for a decreasing *GP*.

A sequence such as 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and so forth is called a *harmonic progression*, or an *HP*. The reciprocals of an *HP* form an *AP*. The general form of an *HP* is

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \frac{1}{a+3d},\ \ldots$$

If the law of formation of a sequence is known, it may be extended to any number of terms.

Exercise 78

Write two additional terms in each of the following:

1. 4, 7, 10, 13	6. 10, 2, .4, .08
2. 4, 2, 1, $\frac{1}{2}$	7. $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{24}$, $\frac{1}{120}$
3. 1, 4, 9, 16	8. −8, −5, −2, 1
4. $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$	9. 8, 12, 18, 27
5. 5, 2, −1, −4	10. −2, 2, $\frac{2}{3}$, $\frac{2}{5}$

95. Arithmetic progression. From the general form of an AP,

$$a, a+d, a+2d, a+3d, \ldots,$$

the 10th term is $a + 9d$, and the nth term is $a + (n - 1)d$. If the nth term is the last term and is designated by l,

$$l = a + (n-1)\,d.$$

The n terms of the AP may be written

$$a, a+d, a+2d, \ldots\; l-2d, l-d, l.$$

The sum, s, of the n terms may be found by noting that the sum of two terms equidistant from the end terms is always $a + l$. Since in n terms there are $\frac{n}{2}$ pairs,

$$s = \frac{n}{2}\,(a+l).$$

The five elements of an AP—namely, a, the first term; l, the last term; n, the number of terms; d, the common difference; and s, the sum of n terms—are related by the equations

$$l = a + (n - 1)\,d$$

and

$$s = \frac{n}{2}\,(a + l).$$

Hence if any three of the five elements are given, the remaining two may be found by solving 2 simultaneous equations containing 2 unknowns.

Illustration

Given $l = 30$, $s = 110$, $d = 4$. Find a and n. The two equations are

$$30 = a + 4(n - 1)$$

and

$$110 = \frac{n}{2}(a + 30).$$

The value $a = 34 - 4n$ from the first equation, when substituted in the second, gives $n^2 - 16n + 55 = 0$, and $n = 11$ or 5. $n = 11$ gives $a = -10$, and the progression is

$$-10, -6, -2, 2, 6, 10, 14, 18, 22, 26, 30.$$

$n = 5$ gives $a = 14$, and the progression is

$$14, 18, 22, 26, 30.$$

Exercise 79

1. Find the remaining elements in each of the following AP's:

(a) $a = 7, d = 3, n = 10$
(b) $a = 7, d = 2, n = 12$
(c) $a = 7, d = -1, n = 6$
(d) $a = 5, l = 75, n = 7$
(e) $a = 3, s = -12, d = -2$
(f) $l = -7, d = -2, s = -12$

2. Mr. Charles borrowed \$500, which he will repay at \$50 per month, beginning one month after the loan was made. He will also pay simple interest at 6% per annum on the unpaid part of the loan when he makes payments on the loan. How much interest will he pay?

3. A debt of \$10,000 is payable in quarterly installments of \$250, beginning 3 months after the loan was made, with interest on the unpaid debt at $1\frac{1}{2}$% per quarter. What is the total amount of interest to be paid?

4. A debt was paid in semiannual installments of \$500, beginning 6 months after the debt was contracted, with interest at $2\frac{1}{2}$% per period of 6 months. The total payments of interest amounted to \$2375. How much was the loan?

5. The 64 squares of a chessboard are numbered consecutively from left to right 1, 2, 3, . . . , 64.

(a) Find the sum of all the numbers.
(b) " " " " " " odd numbers.
(c) " " " " " " even "
(d) " " " " " " numbers on each diagonal.

96. Geometric progression. From the general form of a GP,

$$a, ar, ar^2, ar^3, \ldots,$$

the 10th term is ar^9, and the nth term, l, is

$$l = ar^{n-1}.$$

The sum of n terms, s, is

$$s = a + ar + ar^2 + \ldots + \frac{l}{r^2} + \frac{l}{r} + l.$$

Multiplication by r gives

$$rs = ar + ar^2 + ar^3 + \ldots + \frac{l}{r} + l + lr.$$

These two equations may be rewritten as

$$s = a + (ar + ar^2 + \ldots + \frac{l}{r^2} + \frac{l}{r} + l),$$

$$rs = (ar + ar^2 + \ldots + \frac{l}{r} + l) + lr.$$

Since the terms within the parentheses are the same in the two equations,

$$s - a = rs - lr.$$

Hence

$$s = \frac{a - lr}{1 - r} = \frac{a - ar^n}{1-r} = \frac{a(1 - r^n)}{1-r}.$$

The 5 elements of a GP—namely, a, l, n, r, s—are related by the equations

$$l = ar^{n-1},$$

$$s = \frac{a(1-r^n)}{1-r} = \frac{a-rl}{1-r}.$$

Hence if any three of the five elements are given, the remaining two may be found.

Illustration

Given $a=2$, $l=1458$, $s=2186$. Find r and n.

$l = ar^{n-1}$ gives $1458 = 2r^{n-1}$ or $r^{n-1} = 729$, and $r^n = 729r$.

$s = \dfrac{a(1-r^n)}{1-r}$ gives $2186 = \dfrac{2(1-729r)}{1-r}$, which reduces to $r = 3$.

In $r^{n-1} = 729$, we now have $3^{n-1} = 3^6$, and $n-1 = 6$, or $n = 7$. The progression is

$$2, 6, 18, 54, 162, 486, 1458.$$

Exercise 80

1. In the following *GP*'s find the elements that are not given:

(a) $a = 3,\ r = 2,\ n = 10$	(e) $a = 2,\ l = 486,\ s = 728$
(b) $a = .6,\ r = .4,\ n = 7$	(f) $a = 3,\ n = 5,\ l = 12$
(c) $a = 1,\ r = \frac{1}{2},\ n = 10$	(g) $a = 3,\ l = 12,\ s = 33.73$
(d) $a = 2,\ l = 486,\ n = 6$	(h) $a = 3,\ r = \sqrt{2},\ n = 9$

2. Derive the formula $s_{\overline{n}|i} = \dfrac{(1+i)^n-1}{i}$ from the *GP* $s_{\overline{n}|i} = 1 + s_i^1 + s_i^2 + \ldots$, n terms.

3. Derive the formula $a_{\overline{n}|i} = \dfrac{1-(1+i)^{-n}}{i}$ from the *GP* $a_{\overline{n}|i} = v_i^1 + v_i^2 + v_i^3 + \ldots$, n terms.

97. The mean. The reciprocals of the terms of an *HP* form an *AP*. It is not possible to find a simple formula for the sum of the terms of an *HP*.

If two numbers p and q are given, and a number m is inserted between p and q, then, if p, m, q form an *AP*, m is the *arithmetic mean;* if p, m, q form a *GP*, m is the *geometric mean;* if p, m, q form an *HP*, m is the *harmonic mean.*

(1) If p, m, q form an *AP*, $m - p = q - m$, and $m = \dfrac{q+p}{2}$.

(2) If p, m, q form a *GP*, $\dfrac{m}{p} = \dfrac{q}{m}$, and $m = \sqrt{qp}$.

(3) If p, m, q form an *HP*, $\dfrac{1}{m} - \dfrac{1}{p} = \dfrac{1}{q} - \dfrac{1}{m}$, and $m = \dfrac{2pq}{p+q}$.

Exercise 81

1. Find the arithmetic, the geometric, and the harmonic mean between 2 and 8.

2. Show that the geometric mean between p and q is also the geometric mean between the arithmetic and the harmonic means of p and q.

3. Given the numbers 2 and 8. Insert 3 numbers between them so that the sequence of 5 numbers shall form (a) an AP; (b) a GP; (c) an HP.

4. Three numbers form an AP. If the first is decreased by 1, the second decreased by 2, and the third increased by 1, the resulting numbers form a GP whose common ratio is 2. Find the numbers.

98. Infinite decreasing *GP*. In a GP in which r is numerically less than 1, the terms decrease. The sum

$$s = a\frac{1-r^n}{1-r}$$

may be written

$$s = \frac{a}{1-r} - a\frac{r^n}{1-r}.$$

If r is numerically less than 1, the value of r^n becomes smaller as n is increased, and if n is very large, r^n is very small. Hence if n increases beyond all limits, r^n approaches 0, and the limit that the sum of the series approaches is

$$s = \frac{a}{1-r}.$$

Thus the sum of the series

$$1, \tfrac{2}{3}, \tfrac{4}{9}, \tfrac{8}{27}, \ldots$$

approaches the limit $s = \dfrac{a}{1-r} = \dfrac{1}{1-\frac{2}{3}} = 3.$

Exercise 82

Find the limiting value of the sum of each of the following series

1. $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

2. $2, -1, \frac{1}{2}, -\frac{1}{4}, \ldots$

3. $4, 3, \frac{9}{4}, \frac{27}{16}, \ldots$

4. $4, -3, \frac{9}{4}, -\frac{27}{16}, \ldots$

5. $1, \frac{1}{2}\sqrt{2}, \frac{1}{2}, \frac{1}{4}\sqrt{2}, \ldots$

6. $\sqrt{3}, \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{3}{8}}, \ldots$

99. Circulating decimals. A circulating or repeating decimal such as 4.23333. . . is usually written $4.2\dot{3}$. When more than one figure repeats, dots are placed above the first and the last of the repeating figures. Thus

$$3.2\dot{4}0\dot{7} = 3.2407407407. \ . \ . \ .$$

Every repeating decimal involves an infinite decreasing *GP* and may be changed into an equivalent common fraction. Thus $3.2\dot{4}0\dot{7} = 3.2 + 407 \times 10^{-4} + 407 \times 10^{-7} + 407 \times 10^{-10} + \ . \ . \ . \ .$ The terms after 3.2 form a *GP* in which $a = 407 \times 10^{-4}$, $r = 10^{-3}$, and

$$s = \frac{407 \times 10^{-4}}{1 - 10^{-3}} = \frac{407}{10^4 - 10} = \frac{407}{9990}.$$

Hence

$$3.2\dot{4}0\dot{7} = 3.2 + \frac{407}{9990} = \frac{32375}{9990} = \frac{6475}{1998} = \frac{175}{54}.$$

The common fraction equivalent to a repeating decimal may also be found as follows: The successive digits are a, b, c, d, and so on, and the value of the repeating decimal $ab.c\dot{d}ef\dot{g}$ is represented by x. Multiplying this number by 10^4 (since 4 digits repeat) gives

$$10^4x = abcdef.gdefg. \ . \ . \ ,$$

and

$$x = \qquad ab.cdefg. \ . \ . \ .$$

Subtraction gives

$$x(10^4-1) = abcdef.g-ab.c = \tfrac{1}{10}\,(abcdefg - abc).$$

Hence

$$x = \frac{abcdefg-abc}{10(10^4-1)}.$$

Thus $3.2\dot{4}0\dot{7}$ may be written at once as

$$\frac{32407-32}{10(10^3-1)} = \frac{32375}{9990} = \frac{6475}{1998} = \frac{175}{54}.$$

Exercise 83

Verify the correctness of each of the following equalities:

1. $2.\dot{3} - 1.\dot{4}\dot{5} = .\dot{8}\dot{7}$
2. $2.\dot{3} + 1.\dot{4}\dot{5} = 3.\dot{7}\dot{8}$
3. $2.\dot{3} \times 1.\dot{4}\dot{5} = 3.\dot{3}\dot{9}$
4. $2.\dot{3} \div 1.\dot{4}\dot{5} = 1.6041\dot{6}$
5. $1.\dot{3}0\dot{6} \times 3.\dot{3} = 4.\dot{3}5\dot{0}$
6. $(.\dot{3})^2 = .\dot{1}$

100. Binomial theorem. Ordinary algebraic multiplication gives the results

$$(a+x)^2 = (a+x)\cdot(a+x) = a^2 + 2ax + x^2;$$
$$(a+x)^3 = (a+x)^2\cdot(a+x) = a^3 + 3a^2x + 3ax^2 + x^3;$$
$$(a+x)^4 = (a+x)^3\cdot(a+x) = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

These results may be written:

$$(a+x)^2 = a^2 + \frac{2}{1}a^1x^1 + \frac{2\cdot 1}{1\cdot 2}a^0x^2;$$
$$(a+x)^3 = a^3 + \frac{3}{1}a^2x^1 + \frac{3\cdot 2}{1\cdot 2}a^1x^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}\,a^0x^3;$$
$$(a+x)^4 = a^4 + \frac{4}{1}\,a^3x^1 + \frac{4\cdot 3}{1\cdot 2}\,a^2x^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}\,a^1x^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}\,a^0x^4.$$

The successive terms of each expansion are now seen to obey a simple law, and the value of any positive integral power of $(a+x)$ may be written at sight. Thus

$$(a+x)^{10} = a^{10} + \frac{10}{1}\,a^9x^1 + \frac{10\cdot 9}{1\cdot 2}\,a^8x^2 + \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}\,a^7x^3$$
$$+\frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}a^6x^4 + \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^5x^5 + \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}a^4x^6 + \ldots.$$

In general, if n is a positive integer,

$$(a+x)^n = a^n + \frac{n}{1}\,a^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)\,(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \ldots.$$

If $a = 1$, the expansion takes the useful form

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)\,(n-2)}{1\cdot 2\cdot 3}x^3 + \ldots.$$

Although the expansion of $(1 + x)^n$ was obtained for positive integral values of n, it may be proved that it is likewise true if n is negative or fractional, provided that x is numerically less than 1, whether x is positive or negative.

Illustrations

1. Calculate the value of $\$1000s_{\overline{20}|2\%} = \$1000(1 + .02)^{20}$ to the nearest cent. In the expansion of $(1+x)^n$, $x = .02$ and $n = 20$. Hence

$$1000(1 + .02)^{20} = 1000\left[1+\frac{20}{1}(.02) + \frac{20 \cdot 19}{1 \cdot 2}(.02)^2+ \ldots\right].$$

The values of the successive terms are:

1st term = 1000×1	1000.000
2nd term = 1st term $\times \frac{20}{1} \times .02$	400.000
3rd term = 2nd term $\times \frac{19}{2} \times .02$	76.000
4th term = 3rd term $\times \frac{18}{3} \times .02$	9.120
5th term = 4th term $\times \frac{17}{4} \times .02$	.775
6th term = 5th term $\times \frac{16}{5} \times .02$	.050
7th term = 6th term $\times \frac{15}{6} \times .02$	.003
	1485.948

Note that the value agrees with the value found in Table III. Note also that the value found by logarithms is 1476.

2. Calculate the value of $100(1-.10)^{-20}$.

In the expansion of $(1+x)^n$, $x = -.10$ and $n = -20$. Hence

$$100(1 - .10)^{-20} = 100\left[1 + \left(\frac{-20}{1}\right)(-.10) + \frac{(-20)\ (-21)}{1.2}(-.10)^2 + \frac{(-20)(\ -21)\ (-22)}{1\cdot2\cdot3}(-.10)^3 + \ldots\right]$$

$$= 100\left[1+ \frac{20}{1}(.1) + \frac{20\cdot21}{1\cdot2}(.1)^2 + \frac{20\cdot21\cdot22}{1\cdot2\cdot3}(.1)^3+\ldots\right].$$

The values of the successive terms are:
100, 200, 210, 154, 88.55, 42.504, 17.710, 6.578, 2.220, .691, .200, .055, .014, .003. (Total, 822.53)

Note that the value found by logarithms is 824.20.

Exercise 84

Calculate the values of the following by the binomial expansion:

1. $1000(1-.01)^{30}$
2. $1000(1+.01\frac{1}{3})^{20}$ (write $.01\frac{1}{3}$ as $\frac{4}{300}$, not .0133)
3. $1000(1+.03\frac{1}{3})^{20}$ (write $.03\frac{1}{3}$ as $\frac{1}{30}$, not .0333)
4. $1000(1+.01)^{-30}$
5. $1000(1+.03\frac{1}{3})^{-20}$
6. $1000(1-.03\frac{1}{3})^{-20}$
7. $1000(1+.01)^{30}$
8. $1000(1-.03\frac{1}{3})^{20}$
9. $1000(1-.01)^{-30}$
10. $1000(1-.01\frac{1}{3})^{20}$
11. Calculate $1000s^{20}_{2\%}$ and verify with the table.
12. " $1000v^{20}_{2\%}$ " " " " "
13. " $1000s_{\overline{20}|2\%}$ " " " " "
14. " $1000a_{\overline{20}|2\%}$ " " " " "
15. Use the result of example 3 to calculate $1000s_{\overline{20}|3\frac{1}{3}\%}$.
16. Use the result of example 5 to calculate $1000a_{\overline{20}|3\frac{1}{3}\%}$.
17. Show that $s_{\overline{n}|i} = \frac{n}{1} + \frac{n(n-1)}{1\cdot 2}i + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}i^2 + \ldots.$
18. Show that $a_{\overline{n}|i} = \frac{n}{1} - \frac{n(n+1)}{1\cdot 2}i + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}i^2 - \ldots.$
19. Calculate $(1+i)^{1/p}$ for $i = .02$ and (a) $p = 2$; (b) $p = 3$; (c) $p = 4$; (d) $p = 6$; (e) $p = 12$.
20. Show that $\sqrt[3]{6} = 2(1-\frac{1}{4})^{\frac{1}{3}}$, and calculate its value to 6 decimal places.

101. Difference series. If we take an algebraic polynomial such as $u_x = x^3 - 2x^2 + 5$ and assign values to x at equal intervals, we obtain the table

x	0	1	2	3	4	5	6	7
u_x	5	4	5	14	37	80	149	250

The terms of u_x are designated by u_0, u_1, u_2, u_3, and so on, where $u_3 = 14$ is the value of u_x when $x = 3$.

From the values of u_x, a new set of numbers is made by subtracting in succession u_0 from u_1, u_1 from u_2, and so forth. The new set of numbers is called the *first difference series* and is designated by Δu_x (delta-you-subex), so that $\Delta u_0 = u_1 - u_0 = 4 - 5 = -1$, $\Delta u_1 = u_2 - u_1 = 5 - 4 = 1$, $\Delta u_2 = u_3 - u_2 = 14 - 5 = 9$, and so on.

From the values of Δu_x, a new set of numbers is again made by successive subtractions. This second new set is called the *second difference series* and is designated by $\Delta^2 u_x$ (delta-two-you-subex), so that $\Delta^2 u_0 = \Delta u_1 - \Delta u_0 = 1 - (-1) = 2$, and so on.

Similarly, third, fourth, and higher difference series are designated by $\Delta^3 u_x$, $\Delta^4 u_x$, and so forth.

From the given table, we obtain

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$
0	5	−1	2	6	0
1	4	1	8	6	0
2	5	9	14	6	0
3	14	23	20	6	0
4	37	43	26	6	
5	80	69	32		
6	149	101			
7	250				

102. Expansion of u_x. The subscripts of u_x, Δu_x, and so forth on the first horizontal row are all 0, on the second row, 1, on the third row, 2, and so on.

Because of the method used in making the tabulation, any item in the table is equal to the sum of the one above it and the one to the right of that one. That is, $u_7 = u_6 +$

Δu_6; $\Delta u_6 = \Delta u_5 + \Delta^2 u_5$; $\Delta^2 u_5 = \Delta^2 u_4 + \Delta^3 u_4$; and so on. Hence any term of u_x, say u_4, can be expressed as follows:

(1) $u_4 = u_3 + \Delta u_3$

(2) $u_4 = (u_2+\Delta u_2) + (\Delta u_2+\Delta^2 u_2) = u_2 + 2\Delta u_2 + \Delta^2 u_2$

(3) $u_4 = (u_1+\Delta u_1) + 2(\Delta u_1+\Delta^2 u_1) + (\Delta^2 u_1+\Delta^3 u_1)$
$\quad = u_1 + 3\Delta u_1 + 3\Delta^2 u_1 + \Delta^3 u_1$

(4) $u_4 = (u_0+\Delta u_0) + 3(\Delta u_0+\Delta^2 u_0) + 3(\Delta^2 u_0 + \Delta^3 u_0) + (\Delta^3 u_0 + \Delta^4 u_0)$
$\quad = u_0 + 4\Delta u_0 + 6\Delta^2 u_0 + 4\Delta^3 u_0 + \Delta^4 u_0$

We now observe that the coefficients in the expansions are the same as the coefficients in the expansions of $(a + b)^1$, $(a + b)^2$, $(a + b)^3$, $(a + b)^4$, and that in each expansion the subscripts are the same. That is, if we write u_4 as u_{0+4}, the expansion is in terms of the symbols u, Δu, $\Delta^2 u$, $\Delta^3 u$, all having the subscript 0, and the coefficients are the same as the coefficients in the expansion of the 4th power of $a + b$. Thus

$$u_8 = u_{0+8} = u_0 + \frac{8}{1}\,\Delta u_0 + \frac{8 \cdot 7}{1 \cdot 2}\,\Delta^2 u_0 + \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3}\,\Delta^3 u_0 + \dots,$$

and, in general,

$$u_n = u_0 + \frac{n}{1}\,\Delta u_0 + \frac{n(n-1)}{1 \cdot 2}\Delta^2 u_0 + \frac{n(n-1)\,(n-2)}{1 \cdot 2 \cdot 3}\,\Delta^3 u_0 + \dots.$$

Although this formula was obtained for positive integral values of n, it is equally valid for fractional or negative values of n.

Illustrations

1. $$u_{1/3} = u_{0+1/3} = u_0 + \frac{(\frac{1}{3})}{1}\,\Delta u_0 + \frac{(\frac{1}{3})\,(-\frac{2}{3})}{1 \cdot 2}\,\Delta^2 u_0 + \frac{(\frac{1}{3})\,(-\frac{2}{3})\,(-\frac{5}{3})}{1 \cdot 2 \cdot 3}\,\Delta^3 u_0 + \dots.$$

For the tabulation obtained from $u_x = x^3 - 2x^2 + 5$ (page 175), $u_0 = 5$, $\Delta u_0 = -1$, $\Delta^2 u_0 = 2$, $\Delta^3 u_0 = 6$, $\Delta^4 u_0 = \Delta^5 u_0 = \Delta^6 u_0 = 0$, and $u_{1/3} = 5 + \frac{1}{3}(-1) - \frac{1}{9}(2) + \frac{5}{81}(6) = 4\frac{22}{27}$.

The substitution of $\frac{1}{3}$ for x in $x^3 - 2x^2 + 5$ also gives $4\frac{22}{27}$.

2. $u_{-2} = u_{0-2} = u_0 + \frac{(-2)}{1}\Delta u_0 + \frac{(-2)(-3)}{1\cdot 2}\Delta^2 u_0 +$

$\frac{(-2)(-3)(-4)}{1\cdot 2\cdot 3}\Delta^3 u_0 = 5 - 2(-1) + 3(2) - 4(6) = -11.$

The substitution of $x = -2$ in u_x also gives -11.

3. If we prefer to use the subscript 1 instead of 0 we may write

$$u_{-2} = u_{1-3} = u_1 + \frac{(-3)}{1}\Delta u_1 + \frac{(-3)(-4)}{1\cdot 2}\Delta^2 u_1 + \frac{(-3)(-4)(-5)}{1\cdot 2\cdot 3}\Delta^3 u_1.$$

Since $u_1 = 4$, $\Delta u_1 = 1$, $\Delta^2 u_1 = 8$, $\Delta^3 u_1 = 6$. We then find $u_{-2} = 4 - 3(1) + 6(8) - 10(6) = -11$ as before.

103. Newton's method of interpolation. The formula $u_{0+x} = u_0 + \frac{x}{1}\Delta u_0 + \frac{x(x-1)}{1\cdot 2}\Delta^2 u_0 + \ldots$ is known as *Newton's interpolation formula* and is used as follows: Suppose the values of v_i^n are given for $i = 1\%, 2\%, 3\%, 4\%, 5\%, 6\%$, and the value of $v_{1\frac{1}{2}\%}^{20}$ is required. We write

i	x	$u_x = v_i^{20}$	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$	$\Delta^5 u_x$
1 %	0	.81954	−.14657	+.02728	−.00528	+.00107	−.00024
$1\frac{1}{2}$%							
2 %	1	.67297	−.11929	+.02200	−.00421	+.00083	
3 %	2	.55368	−.09729	+.01779	−.00338		
4 %	3	.45639	−.07950	+.01441			
5 %	4	.37689	−.06509				
6 %	5	.31180					

The relation for the table of corresponding values of i and x is linear, and $i = 1\frac{1}{2}\%$ corresponds to $x = \frac{1}{2}$. Therefore

$$v_{1\frac{1}{2}\%}^{20} = u_{1/2} = u_0 + \frac{\frac{1}{2}}{1}\Delta u_0 + \frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}\Delta^2 u_0 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}\Delta^3 u_0 + \ldots$$

$$= u_0 + \tfrac{1}{2}\Delta u_0 - \tfrac{1}{8}\Delta_2 u_0 + \tfrac{1}{16}\Delta^3 u_0 - \tfrac{5}{128}\Delta^4 u_0 + \tfrac{7}{256}\Delta^5 u_0 + \ldots$$

$$= .81954 + \tfrac{1}{2}(-.14656) - \tfrac{1}{8}(.02728) + \tfrac{1}{16}(-.00528)$$

$$- \tfrac{5}{128}(.000107) + \tfrac{7}{256}(-.00024)$$

$$= .81954 - .073280 - .003410 - .000330 - .000042 - .000006$$
$$= .742472.$$

The value of $v^{20}_{1\frac{1}{2}\%}$ in Table IV is .74247.

If only the first difference, Δu_0, is used, the result, $u_0 + \frac{1}{2}\Delta u_0$, is a first approximation and is the same as a linear interpolation. If the second difference, $\Delta^2 u_0$, is also used, the result, $u_0 + \frac{1}{2}\Delta u_0 - \frac{1}{8}\Delta^2 u_0$, is a second approximation. Similarly, a third, fourth, or fifth approximation may be found, the result being more nearly accurate for each higher approximation.

For most mathematical tables in common use, the first term of the third difference series is smaller than that of the second difference series and so on. Therefore the successive corrections are smaller and smaller, and Newton's interpolation formula is of great practical use.

The tabulation used for calculating $v^{20}_{1\frac{1}{2}\%}$ may also be used for the calculation of $v^{20}_{1\frac{1}{3}\%}$ by setting $x = 1\frac{1}{3}$.

However, tables should always be used to the best advantage, and, since the intervals for i in the table for v^n_i are $\frac{1}{2}\%$, the correspondence, for $i = 2\frac{1}{3}\%$, for example, is

i	2%	$2\frac{1}{3}\%$	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	4%
x	0	$\frac{2}{3}$	1	2	3	4

and $v^{20}_{2\frac{1}{3}\%}$ is found more accurately and with less work from $u_{2/3}$ for this tabulation.

If $v^{20}_{3\frac{7}{8}\%}$ is required, write the values of i in reverse order

i	4%	$3\frac{7}{8}\%$	$3\frac{1}{2}\%$	3%	$2\frac{1}{2}\%$	2%
x	0	$\frac{1}{4}$	1	2	3	4

and find $u_{1/4}$ for this tabulation.

This tabulation may also be used to find the value of $v^{20}_{4\frac{1}{4}}$ by setting $x = -\frac{1}{2}$ and calculating $u_{-1/2}$.

Exercise 85

1. Calculate the value of $1000s^{20}_{1\frac{1}{3}\%}$ (a) by using the binomial expansion; (b) by using the tables and interpolating to a third approximation.

2. Calculate the value of $10a_{\overline{20}|3.2\%}$ by interpolating to a third approximation. Verify the result by finding the value of the symbol from the bond table.

3. Assuming that the bond table, Table X, gives correct prices for the yield rates 3, 4, 5, 6, 7, 8, 9, and 10%, find by interpolating to a third approximation the prices for the yield rates (a) 3.2%; (b) 4.4%; (c) 5.6%; (d) 6.8% for each of the coupon rates tabulated.

104. Continuous increase. The total accumulated value of \$100 at the end of 10 years, if interest is at 4% per annum compounded annually, semiannually, quarterly, or monthly, is given by

$$100(1+.04)^{10},\ 100\left(1+\frac{.04}{2}\right)^{10\times 2},\ 100\left(1+\frac{.04}{4}\right)^{10\times 4},\ \text{or}\ 100\left(1+\frac{.04}{12}\right)^{10\times 12}.$$

The more frequently interest is compounded, the smaller is the rate per period and the larger is the number of periods. We thus arrive at the conception of interest being compounded continuously when the rate per period is very small and the number of periods in a year is very large.

The idea of continuous increase is useful practically, for, as we shall find presently, the calculation for continuous increase is much simpler than for compounding, say, weekly or daily. The difference between the two results is very small.

Let c be invested for n years and let interest at i per annum be compounded k times a year. The accumulated value is $c\left(1+\frac{i}{k}\right)^{nk}$, which, when expanded, gives

$$c\left[1+\frac{nk}{1}\left(\frac{i}{k}\right)+\frac{(nk)(nk-1)}{1\cdot 2}\left(\frac{i}{k}\right)^2+\frac{(nk)(nk-1)(nk-2)}{1\cdot 2\cdot 3}\left(\frac{i}{k}\right)^3+\cdots\right].$$

This result is simplified and written

$$c\left[1+\frac{ni}{1}+\frac{n\left(n-\frac{1}{k}\right)}{1\cdot 2}i^2+\frac{n\left(n-\frac{1}{k}\right)\left(n-\frac{2}{k}\right)}{1\cdot 2\cdot 3}i^3+\cdots\right].$$

As the compounding is more frequent, k is larger. When k increases beyond all bounds or approaches ∞, the fractions, $1/k$, $2/k$, $3/k$, and so on, approach 0 and the expansion reduces to

$$c\left[1 + \frac{ni}{1} + \frac{(ni)^2}{1 \cdot 2} + \frac{(ni)^3}{1 \cdot 2 \cdot 3} + \cdots\right].$$

When k approaches ∞, $c\left(1 + \frac{i}{k}\right)^{nk}$ also reduces to a simple form.

The expression $(1 + \frac{1}{z})^z$, where the exponent z is the reciprocal of the term added to 1, has a value when z approaches ∞, and that value is designated by the symbol e. Now let

$$\frac{i}{k} = \frac{1}{z};$$

then

$$k = iz,\ nk = niz,$$

and

$$c\left(1 + \frac{i}{k}\right)^{nk} = c\left(1 + \frac{1}{z}\right)^{niz} = c\left[\left(1 + \frac{1}{z}\right)^z\right]^{ni}.$$

But when k approaches ∞, z does also, and this expression becomes ce^{ni}.

Hence if c is invested for n years and interest at i per annum is compounded continuously, the total value is given by

$$ce^{ni} = c\left[1 + \frac{ni}{1} + \frac{(ni)^2}{1 \cdot 2} + \frac{(ni)^3}{1 \cdot 2 \cdot 3} + \cdots\right].$$

If we set $ni = h$, the relation

$$e^h = 1 + \frac{h}{1} + \frac{h^2}{1 \cdot 2} + \frac{h^3}{1 \cdot 2 \cdot 3} + \cdots$$

enables us to calculate any power of e. The value of e is found by making $h = 1$. Then

$$e = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$
$$= 2.718. \ldots$$

Thus \$100 invested for 10 years at 4% per annum compounded continuously amounts to

$$100e^{10 \times .04} = 100\left[1 + \frac{.4}{1} + \frac{(.4)^2}{1 \cdot 2} + \frac{(.4)^3}{1 \cdot 2 \cdot 3} + \cdots\right] = 149.182.$$

Note the following results of investing \$100 for 10 years at 4% per annum and compounded:

(a)	annually, total........................	\$148.024
(b)	semiannually, total....................	148.595
(c)	quarterly, total.......................	148.886
(d)	monthly, total.........................	149.083
(e)	continuously, total....................	149.182

The principle of equivalent interest rates enables us to find the corresponding values of $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ when interest is compounded continuously.

Illustrations

1. Forty quarterly deposits of \$100 each are made. Find the total value immediately after the last deposit if interest is compounded continuously at 4% per annum.

The value of the annuity, S, is

$$S = 100s_{\overline{40}|i} = 100\frac{(1+i)^{40}-1}{i},$$

where i = the rate for 3 months.

The relation for equivalent interest rates is $(1+i)^4 = e^{.04}$ and

$$S = 100\frac{e^{.4}-1}{e^{.01}-1}.$$

The values of $e^{.4}$ and of $e^{.01}$ may be calculated from

$$e^h = 1 + \frac{h}{1} + \frac{h^2}{1 \cdot 2} + \frac{h^3}{1 \cdot 2 \cdot 3} + \cdots,$$

or they may be taken from Table XI.

2. The value of the annuity of illustration 1 three months before the first deposit is

$$100a_{\overline{40}|i} = 100\,\frac{1-(1+i)^{-40}}{i} = 100\,\frac{1-e^{-.4}}{e^{.01}-1}.$$

105. Continuous decrease. An investment increases in value because of interest earned. A machine or a building *decreases* in value because of depreciation. If the decrease for any period is always the same per cent, d, of the value at the beginning of the period, then, if the initial value is c, and the decrease continues for n periods giving a final value s, the relation is $s = c(1 - d)^n$. The formula may be obtained from $c(1 + i)^n$ by merely replacing i by $-d$ (see also page 187).

Similarly, the formula for continuous decrease may be obtained from the formula for continuous increase by merely replacing i by $-d$.

Since an investment of c increasing continuously for n years at i per annum gives a final value of ce^{ni}, the result of decreasing continuously at d per annum is ce^{-nd}.

Exercise 86

1. The population of a town increased continuously for 10 years from 50,000, at 2% per annum. Find the population at the end of 10 years.

2. The population of a town increased continuously for 10 years from 50,000 to 65,000. Find the annual rate of increase.

3. A machine decreased in value continuously for 8 years at 10% per annum. Find the final value if the initial value was \$1000.

4. A machine decreased in value continuously for 8 years from \$1000 to \$250. Find the annual rate of decrease.

5. A quantity of gas increased in volume continuously from 1000 cu. ft. at 4% per hour for 5 hours and then decreased continuously for 8 hours until its volume was again 1000 cu. ft. Find the hourly rate of decrease.

6. Find the value of a \$1000 bond maturing at the end of 10 years and paying interest semiannually at 4% per annum if the yield rate is 6% per annum compounded continuously.

7. Find the amount of each of 20 quarterly deposits if the total immediately after the last deposit is \$1000, and interest is compounded continuously at 4% per annum.

8. In the following symbols, the periods are 6 months each and the rate of interest is 4% per annum compounded continuously. Find the value of each: (a) $1000s^{20}$; (b) $1000v^{30}$; (c) $100a_{\overline{20}|}$; (d) $100s_{\overline{30}|}$.

106. Logarithmic series. From the relation

$$y = e^x - 1 = x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots,$$

it is easy enough to calculate y for any value of x. To calculate x for a given value of y, it is necessary to express x in terms of y. Since $x = 0$ gives $y = 0$, we assume that the relation has the form

$$x = y + Ay^2 + By^3 + Cy^4 + \cdots.$$

Then

$$\frac{x^2}{1 \cdot 2} = \tfrac{1}{2}y^2 + Ay^3 + \left(B + \frac{1}{2}A^2\right)y^4 + \cdots$$

$$\frac{x^3}{1 \cdot 2 \cdot 3} = \frac{1}{6}y^3 + \frac{1}{2}Ay^4 + \cdots$$

$$\frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{1}{24}y^4 + \cdots.$$

Addition gives for the second member

$$y + y^2(A + \tfrac{1}{2}) + y^3(B + A + \tfrac{1}{6}) + y^4(C + B + \tfrac{1}{2}A^2 + \tfrac{1}{2}A + \tfrac{1}{24}) + \cdots.$$

In order that this sum shall reduce to y, the sum of the first members, it is necessary that

$$A + \tfrac{1}{2} = 0,\ B + A + \tfrac{1}{6} = 0,\ C + B + \tfrac{1}{2}A^2 + \tfrac{1}{2}A + \tfrac{1}{24} = 0,$$

or that

$$A = -\tfrac{1}{2},\ B = \tfrac{1}{3},\ C = -\tfrac{1}{4}.$$

The series is

$$x = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \cdots,$$

which may be extended to any number of terms.

The relation $y = e^x - 1$ gives $1 + y = e^x$ and $\log_e (1 + y) = x$. Hence

$$\log_e (1 + y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \tfrac{1}{5}y^5 - \ldots.$$

Upon replacing y by $-y$, the new relation is obtained

$$\log_e (1 - y) = -y - \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 - \tfrac{1}{5}y^5 - \ldots$$

Then

$$\log_e (1 + y) - \log_e (1 - y) = 2(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \ldots),$$

or

$$\log_e \frac{1 + y}{1 - y} = \log_e N = 2(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \ldots).$$

107. Napierian logarithms. The logarithmic series

$$\log_e \frac{1 + y}{1 - y} = \log_e N = 2(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \ldots)$$

enables us to calculate $\log_e N$, the Napierian logarithm of N, very rapidly by giving small fractional values to y. Thus if $y = \frac{1}{10}$, $N = \frac{11}{9}$, and

$$\log_e \tfrac{11}{9} = 2\,[(.1) + \tfrac{1}{3}\,(.1)^3 + \tfrac{1}{5}\,(.1)^5 + \ldots] = .200{,}670{,}695^+.$$

The values

y	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{20}$	$\frac{1}{25}$	$\frac{1}{50}$
N	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{11}{9}$	$\frac{8}{7}$	$\frac{21}{19}$	$\frac{13}{12}$	$\frac{51}{49}$

enable us to calculate the logarithms of the prime numbers 2, 3, 5, 7, 11, 13, 17, 19, and therefore the logarithms of the numbers from 1 to 20.

The student may verify the results

$$\log_e \tfrac{3}{2} = 0.405{,}465{,}108;\quad \log_e \tfrac{5}{4} = 0.223{,}143{,}551;$$
$$\log_e \tfrac{4}{3} = 0.287{,}682{,}072;\quad \log_e \tfrac{8}{7} = 0.133{,}531{,}393.$$

Then

$$\log 2 = \log \tfrac{3}{2} + \log \tfrac{4}{3} = \log (\tfrac{3}{2} \times \tfrac{4}{3}) = 0.693{,}147{,}180;$$
$$\log 3 = \log \tfrac{3}{2} + \log 2 = \log (\tfrac{3}{2} \times 2) = 1.098{,}612{,}288;$$
$$\log 5 = \log \tfrac{5}{4} + \log 2^2 = \log (\tfrac{5}{4} \times 4) = 1.609{,}437{,}912.$$

Table XII is a table of Napierian logarithms.

108. Common logarithms. From the Napierian logarithms it is a simple matter to calculate the common logarithms. Let $\log_{10} N = x$. Then

$$N = 10^x, \log_e N = x \log_e 10, x = \frac{\log_e N}{\log_e 10}.$$

That is, x, or $\log_{10} N$, is found by dividing $\log_e N$ by $\log_e 10$, or by multiplying $\log_e N$ by

$$\frac{1}{\log_e 10} = \frac{1}{2.302,585,093} = 0.434,294,48.$$

The method of abbreviated multiplication, page 120, gives

N	$\log_e N$	$\log_{10} N$
1	0	0
2	0.693,147,180	0.301,029,99
3	1.098,612,288	0.477,121,25
4	1.386,294,360	0.602,059,98
5	1.609,437,912	0.698,970,01
6	1.791,759,468	0.778,151,24
7	1.945,910,147	0.845,098,04
8	2.079,441,540	0.903,089,97
9	2.197,224,576	0.954,242,50
10	2.302,585,093	1.000,000,00

Note: The student should now be able to solve any of the problems presented in Art. 75, page 126.

CHAPTER IX

VALUATION OF FIXED ASSETS

109. Depreciation. Machines and buildings generally decrease in value or *depreciate* as a result of use or merely because of the lapse of time. When such an asset is used for the purpose of producing income, some of the income is set aside at regular intervals as a reserve to provide for the replacement of the asset when it has outworn its usefulness. Past experience enables us to estimate the useful life of the asset and also its *final*, *residual*, or *scrap value*. If the initial cost is c, and the scrap value is s, which is also the *book value* at the end of n years, the loss in value is $c - s$.

Some of the many methods of apportioning the loss $c - s$ are: (a) the straight line method, (b) the units' digit method, (c) the reducing balance method, (d) the sinking fund method, (e) the annuity method, (f) the unit production method, (g) the number of hours used method.

In the following discussions, we shall see how some of these methods are applied in the case of a machine that cost \$1,000 and has a scrap value of \$50 at the end of 10 years.

110. Straight line method. If it is assumed that the loss in value is the same each year, the annual depreciation is $\frac{c-s}{n}$. At the end of x years, the loss is $\frac{c-s}{n}x$, and the book value, y, is $y = c - \frac{c-s}{n}x$. This equation is represented graphically by a straight line.

For our problem the annual depreciation is

$\frac{1}{10}$ (\$1000 $-$ 50) $=$ \$95 and the equation is $y = 1000 - 95x$.

The straight line method of calculating depreciation is commonly used because of its arithmetic simplicity. It is open to the objection, however, that, as the machine ages, more money must be spent for repairs. The actual cost of operating the machine increases from year to year whereas it is desirable to apportion the depreciation so as to make the annual cost approximately constant.

111. Units' digit method. To overcome the objection raised to the straight line method, the numbers of the years are added and the depreciation for the different years are determined for our problem as follows: $1 + 2 + 3 + \ldots + 9 + 10 = 55$. For the first year the depreciation is $\frac{10}{55}$ of \$950 $=$ \$172.73; for the second year, $\frac{9}{55}$ of \$950, or $\frac{9}{10}$ of \$172.73; for the third year, $\frac{8}{55}$ of \$950, or $\frac{8}{10}$ of \$172.73; and so on.

112. Reducing balance method. It is assumed that each year the decrease in value is the same fraction of the book value at the beginning of the year. Thus if the fraction decided upon is $\frac{1}{6}$ or $16\frac{2}{3}\%$, the depreciation for the first year is $\frac{1}{6}$ of \$1000, leaving a book value of $\frac{5}{6}$ of \$1000. For the second year the depreciation is $\frac{1}{6}$ of ($\frac{5}{6}$ of \$1000), and the book value at the end of the second year is $\frac{5}{6}$ of ($\frac{5}{6}$ of \$1000), or $(\frac{5}{6})^2 \times$ \$1000. At the end of the third year the book value is $(\frac{5}{6})^3 \times$ \$1000, and so forth.

In general, if d is the rate of decrease, the depreciation at the end of the first year is cd, leaving a book value of $c - cd$, or $c(1 - d)$. For the second year the depreciation is $cd\,(1 - d)$, leaving a book value of $c(1 - d) - cd(1 - d) = c(1 - d)^2$. At the end of the third year the book value is $c(1 - d)^3$, and at the end of the nth year it is $c(1 - d)^n$. But since this book value is also the scrap value, the relation is $c(1 - d)^n = s$.

In our problem the equation becomes

$$1000\ (1-d)^{10} = 50,$$

or

$$(1-d)^{10} = .05.$$

The value of d, found by using logarithms, is .2588, or 25.88%.

113. Comparison of the results of the three methods. The following tabulation shows the depreciation for each year and the book value at the end of each year by the three methods:

Year	Depreciation for the Year			Book Value at End of Year		
	Straight Line	*Digits*	*Reducing Balance*	*Straight Line*	*Digits*	*Reducing Balance*
1	$95	$172.73	$258.80	$905	$827.27	$741.20
2	95	155.45	191.82	810	671.82	549.38
3	95	138.18	142.18	715	533.64	407.20
4	95	120.91	105.38	620	412.73	301.82
5	95	103.64	78.11	525	309.09	223.71
6	95	86.36	57.90	430	222.73	165.81
7	95	69.09	42.91	335	153.64	122.90
8	95	51.82	31.81	240	101.82	91.09
9	95	34.55	23.57	145	67.27	67.52
10	95	17.27	17.47	50	50.00	50.05

Exercise 87

1. Draw three graphs with reference to the same axes for the tabulation, marking the number of years along the x axis and the book values along the y axis.

2. The cost of a machine is $2000, and it is estimated that the scrap value at the end of 12 years will be $150.

(a) Find the annual depreciation by the straight line method.

(b) Find the amount of depreciation to be written off for the 12th year by the units' digit method.

(c) Find the annual rate of depreciation to be used by the reducing balance method.

3. When the reducing balance method is used, the relation $c(1-d)^n = s$ enables us to calculate the value of any one of the four symbols c, d, n, or s, provided the remaining three are known.

Solve the equation for each of the four symbols.

4. Fill in the blanks in the following tabulation by using logarithms:

	c	d	n	s
(a)	1000	10%	15 years	....
(b)	1000	10%	....	$200
(c)	....	10%	15 years	250
(d)	1000	....	15 years	200

(e) Calculate the value of s in example (a) by using the binomial expansion.

(f) Calculate the value of c in example (c) by using the binomial expansion.

(g) In example (b), find the depreciation for each of the first three years and also for each of the last three years without finding the life of the machine.

5. When reducing balance method of depreciation is used, the book values x and y of a machine at the end of n and m years are

$$x = c(1-d)^n \text{ and } y = c(1-d)^m.$$

Show by eliminating d from these equations that the relation is

$$\left(\frac{x}{c}\right)^m = \left(\frac{y}{c}\right)^n.$$

6. Use the relation of example 5 to find the book value of a machine at the end of 7 years if the cost was $1000 and the scrap value at the end of 15 years will be $150, without finding the constant rate.

114. Sinking fund method. Provision for the replacement of a machine at original cost when the machine is ready to be scrapped may be made by depositing equal amounts, x, at the end of each of the n years of the life of the machine, into a fund that earns interest at rate i per annum. The total that will be available at the end of n years is $xs_{\overline{n}|i} + s$, and, as this total should be equal to c,

$$xs_{\overline{n}|i} + s = c,$$

and

$$x = (c-s) \cdot \frac{1}{s_{\overline{n}|i}}.$$

Thus in the case of the machine that we have been discussing, if the fund earns interest at 4% per annum,

$$x = 950 \cdot \frac{1}{s_{\overline{10}|4\%}} = \$79.13.$$

At the end of 10 years the fund will amount to $\$79.13 s_{\overline{10}|4\%}$ = $950.04, which, with the scrap value of $50, will give enough to buy a new machine.

The depreciation for any year is usually considered to be the amount deposited to the fund and the interest earned by the fund during that year. Thus the depreciation charged for any year is greater than for the preceding year. If the method of writing off depreciation is to make the annual cost of operating the machine approximately constant, the sinking fund method is not even so good as the straight line method.

115. Annuity method. By this method it is assumed that the initial investment in the machine should earn interest at the same rate as the sinking fund. The annual loss, x, then becomes

$$x = (c-s) \cdot \frac{1}{s_{\overline{n}|i}} + ci,$$

which in the case of the machine discussed is $79.13 + 40 = $119.13.

The annuity method is sometimes stated as follows: Find how much each annual payment, y, should be, so that the value of the annuity at the date of the purchase of the machine shall be the cost less the value of the scrap at that date. The relation then is

$$y a_{\overline{n}|i} = c - v_i^n s,$$

or

$$y = (c - v_i^n s) \cdot \frac{1}{a_{\overline{n}|i}}.$$

The student may show algebraically that the values of x and y are identical.

The depreciation written off for any year by this method is x or y. For the machine discussed, the depreciation written off each year is $119.13.

The book value of the machine at the end of any year is exactly the same as that obtained by using the sinking fund method.

Exercise 88

A machine costs $2000 and it is estimated that at the end of 8 years its value will be $250. If interest is at 3% per annum effective, find:

1. The annual deposit necessary into a sinking fund.

2. The amount of depreciation to be written off for each of the 8 years by the sinking fund method.

3. The amount of depreciation to be written off for each of the 8 years by the annuity method.

116. Capitalized cost. Questions such as the following often arise:

(a) Which is more economical, to buy a machine that costs $50 and will have a life of 10 years or one that costs $75 and will have a life of 15 years?

(b) How much can we afford to spend on improvements on a machine that costs $100 and will have a life of 10 years, if the improvements will extend the life to 15 years?

In either case, since an investment is to be made for a long time, interest must be taken into account. It is assumed that either machine will be satisfactory for the use to which it will be put.

A simple method of making comparisons is to find the *capitalized cost* of each machine. Capitalized cost means the amount that is theoretically necessary to buy the machine at a fixed price and to continue replacing it by

a new one forever. The replacement cost is also fixed and may or may not be the same as the original cost. The word "forever" may appear to complicate the problem but really simplifies it.

Illustrations

1. A machine costs \$50, will have a life of 10 years, and will be replaced at original cost. Find the capitalized cost if interest is at 3% per annum.

Let the capitalized cost be \$50 + x, of which \$50 will be used immediately and x will be invested at 3% per annum for 10 years.

The annual interest earned, $.03x$, constitutes an annuity whose total value at the end of 10 years will be $.03xs_{\overline{10}|3\%}$, and this total should be \$50, sufficient to pay for a new machine. Hence

$$x = \frac{50}{.03} \times \frac{1}{s_{\overline{10}|3\%}} = \$145.39$$

That is, \$195.39 is needed to buy the machine and to replace it by a new one forever at original cost, for \$145.39 invested for 10 years at 3% will amount to $\$145.39s_{3\%}^{10} = 195.39$ when \$50 will be spent, leaving \$145.39 for investment, and the process may be continued forever.

It may be noted that if the replacement cost is not \$50 but \$40, or 80% of the original cost, the capitalized cost is \$50 + 80% of 145.39 = \$166.31.

2. The machine of example 1 can have its life extended to 16 years. How much can we afford to spend for this purpose?

The capitalized cost of the new machine should not be greater than that of the old one. Then if y is the cost of the new machine and x is to be invested, the value of the investment at the end of 16 years will be $xs_{3\%}^{16}$ when y will be spent, and x should again remain invested. Hence

$$xs_{3\%}^{16} - y = x,$$

and

$$xs_{3\%}^{16} = x + y = 195.39,$$

or

$$x = 195.39v_{3\%}^{16} = 121.76.$$

Therefore $y = 195.39 - 121.76 = 73.63$, or the new machine will cost \$23.63 more than the old one.

If the improvements can be gotten for less than \$23.63, it is economical to make the improvements.

Exercise 89

1. Which is more economical: (a) an article that costs \$100 and will have to be replaced each year; or (b) an article that will serve the same purpose but costs \$300 and will have to be replaced every 3 years? Interest is at 6% per annum effective.

2. In example 1, change 3 years to 4 years and solve the problem.

3. An article costs \$10 and will have to be replaced every 5 years. A chemical process that adds \$5 to the cost extends the life of the article to 10 years. If interest is at 5% per annum effective, is it economical to subject the article to the chemical treatment?

4. What is the highest additional cost that can be considered in example 3?

5. A wealthy man wishes to endow a university with a fund for certain equipment that costs \$15,000 and that will have to be replaced by new equipment at the same cost at the end of every 8 years. If the fund can be invested at 4% per annum, convertible quarterly, how large a fund is required?

6. If in example 5 the first cost is \$20,000 and replacements are required at the end of every 10 years at a cost of \$10,000, how large a fund is necessary?

7. A machine will cost \$10,000 to install and at the end of 10 years it will be replaced by a new machine which will cost \$5000. How much should be deposited into a fund paying interest at 4% per annum to provide for the replacement of the machine at the end of every 10 years forever?

8. A company sets aside \$10,000 for the capitalized cost of a truck which costs \$1000 and must be replaced every 3 years with a new truck at the same price, the scrap value of the old truck being \$100. Find the rate of interest that the investment earns if the capital remains intact.

9. A company is to produce a new floor covering which, tests show, will wear for 3 years before it requires replacement under normal

wear. The company wishes to price the covering so that it can compete, on an equal price basis, with a company selling a rival floor covering which lasts 2 years and sells for $1 per square yard. What price should be set for the new covering if money is considered to be worth 4% per annum effective?

10. Fifty thousand dollars is available to equip a laboratory at a cost of $20,000, and to provide for replacement of equipment, at a cost of $14,000, every 10 years. What effective rate of interest must be earned on all invested moneys so that the $50,000 is adequate for the purposes mentioned?

11. A certain type of pavement costs $12 per unit area and must be renewed at the same cost at the end of every 10 years. How much could a highway commission afford to pay for a different variety of pavement if the latter required renewal at its initial cost at the end of each period of 15 years? Money is worth 4% per annum compounded annually.

12. A building costing $4000 must be rebuilt every 20 years. What sum, set aside when the building is erected, will provide for its perpetual replacement, assuming that the cost of rebuilding will always be $4000 and that interest will be at 4% per annum compounded annually?

13. An article costs $8 and has a useful life of 4 years. A second article serves the same purpose and has a useful life of 8 years. If money is worth 5% per annum effective, what price should be paid for the second article?

CHAPTER X

LIFE INSURANCE

117. Insurance. All insurance companies do business in very much the same way. In consideration of a small annual payment (*premium*), the insurance company prepares a contract (*policy*) which provides for the payment of a large amount (*face of policy*) to some individual (*the beneficiary*) in the event that the *contingency* (fire, theft, illness, or death) occurs to the specified property or to the person whose health or life is considered (*the insured*).

Property and accident insurance companies are operated for the benefit of a relatively small number of stockholders, who provide the money with which the company carries on its business.

Life insurance companies are generally operated for the mutual benefit of the policyholders, who are the only stockholders in the company and who share in the profits. Thus, life insurance companies are mutual benefit associations. Many of the safeguards against fire, accident, disease, and death which most modern communities enjoy are the result of the combined efforts of insurance companies and their policyholders.

The premium charged by an insurance company, usually at so many dollars per $1000 of insurance, is the result of calculations based upon statistics of past experience.

118. Mortality table. The record of past experience used by life insurance companies in the United States is the American Experience Table of Mortality, Table XIV, prepared in 1868 from data compiled by life insurance companies, health boards, and other agencies.

Briefly described, it is a summary of the life records of 100,000 normal, healthy children 10 years of age. Because of deaths from any cause whatsoever, the number diminishes year by year until, at the age of 95, only 3 of the original 100,000 are still alive, and none reach the age of 96.

The mortality table consists of 3 columns headed, respectively, x, l_x, and d_x.

Column x lists ages from 10 to 95. Column l_x lists the number alive at age x. Thus $l_{10} = 100{,}000$ and $l_{40} = 78{,}106$ mean that, of the group considered, 100,000 are alive at age 10 and that, of this group, only 78,106 reach the age of 40. Column d_x lists the number that die between ages x and $x + 1$. Thus $d_{30} = 720$ means that, of the group considered, 720 die after 30, but before reaching age 31. Hence $d_{30} = l_{30} - l_{31}$. The total number that die after reaching 30 and before reaching 40, $d_{30} + d_{31} + \ldots + d_{39}$, is indicated by writing $\sum\limits_{30}^{39} d_x$, read "sigma d_x from $x = 30$ to $x = 39$." (The symbol Σ is discussed in greater detail on page 234.)

Exercise 90

1. From the meanings of the symbols and without reference to the table, show that:

$$\text{(a) } \sum_{10}^{13} d_x = l_{10} - l_{14}; \text{ (b) } \sum_{30}^{50} d_x = l_{30} - l_{51}; \text{ (c) } \sum_{50}^{95} d_x = l_{50}.$$

2. Verify the equalities in example 1 by the table.

3. Given $l_{40} = 78{,}106$ and $d_{40} = 765$. What additional entry can you calculate?

4. What additional entry can you calculate from $l_{30} = 85{,}441$ and $l_{31} = 84{,}721$?

5. Indicate by means of proper symbols the number that die between ages 25 and 35.

6. Take the values of l_{30}, l_{35}, l_{40}, l_{45}, l_{50} from Table XIV, and find the value of l_{31} by using Newton's method of interpolation.

119. Standard policies. A life insurance policy states specifically the obligations of the policyholder and of the company. The following are some of the usual or *standard* forms. It is understood that premium payments cease upon the death of the insured.

1. *Ordinary life.* This policy requires that the insured pay equal premiums annually as long as he lives. The insurance company agrees to pay the face of the policy to the beneficiary upon the death of the insured.

2. *Twenty payment life.* This policy requires that the insured pay 20 equal annual premiums. The company agrees to pay the face of the policy to the beneficiary upon the death of the insured, whether death occurs before or after 20 premiums have been paid.

3. *Twenty year endowment.* This policy requires that the insured pay 20 equal annual premiums. The company agrees to pay the face of the policy at the end of 20 years if the insured is still alive. It also agrees to pay the face of the policy if the insured dies before the expiration of 20 years.

4. *Ten year term.* This policy requires that the insured pay 10 equal annual premiums. The company agrees to pay the face of the policy if the insured dies before the expiration of 10 years. If the insured is alive after the expiration of 10 years, the company does not accept any further premiums, nor does it assume any liability in case of subsequent death. A 10 year term policy may be converted into another form of policy at any time during the first 7 years that it is in force, without a physical examination.

Policies are issued similar to 2, 3, or 4 but specifying a different number of payments. Premium payments on any policy may be made in semiannual or quarterly installments which include a small charge for interest.

120. Commutation columns. To calculate the annual premium on a life insurance policy taken at age x it is assumed:

1. Every one of the l_x individuals of the group takes a policy having the same provisions as the policy considered.
2. Premium payments are made at the *beginning* of each policy year.
3. Death claims are paid by the company at the *end* of the policy year during which death occurs.

The calculation of the premium is an application of the principle of equivalent obligations—that is, the value at any date of the obligations of the policyholder is exactly equal to the value at that date of the obligations of the company. Calculations are simplified if the date of comparison selected is the date of birth of the insured.

Thus \$1 paid by each of l_{30} policyholders has a value at the date of birth, $l_{30}v_i^{30}$, and \$1 paid by the company for each of d_{30} deaths has a value at the date of birth, $d_{30}v_i^{31}$, since death claims are paid at the end of the year. The column headed v^x is merely v_i^n where $n = x$ and $i = 3\frac{1}{2}\%$, not written but understood. The column headed l_xv^x designated by D_x is obtained by multiplying l_x by v^x. Thus $D_{30} = l_{30}v^{30} = 85{,}441 \times .35628 = 30{,}441$ to the nearest unit.

Column N_x is the sum of the entries in the column D_x to the end of the table. Thus $N_{30} = \sum_{30}^{95} D_x = D_{30} + D_{31} + \ldots + D_{95} = 596{,}804$. Any other summation, say from $x = 30$

to $x = 42$, is indicated by $\sum_{30}^{42} D_x$. Its value is $N_{30} - N_{43}$ since

$$D_{30} + D_{31} + \ldots + D_{42} = (D_{30} + \ldots + D_{42}) + (D_{43} + \ldots + D_{95}) \\ - (D_{43} + \ldots + D_{95}).$$

Column $C_x = d_x v^{x+1}$ is obtained by multiplying d_x by v^{x+1}. Thus $C_{30} = d_{30}v^{31} = 720 \times .34423 = 248$ to the nearest unit.

Column M_x is the sum of the entries in the column C_x to the end of the table.

$$M_{30} = C_{30} + C_{31} + \ldots + C_{95} = 10{,}259.$$

$$\begin{aligned} \sum_{30}^{49} C_x &= C_{30} + \ldots + C_{49} = (C_{30} + \ldots + C_{95}) - (C_{50} + \ldots + C_{95}) \\ &= M_{30} - M_{50}. \end{aligned}$$

The columns D_x, N_x, C_x, M_x are called *commutation columns*. In our table they are calculated for an interest rate of $3\frac{1}{2}\%$ per annum. Similar tables are obtainable for other interest rates.

121. Net annual premium. The annual premium calculated so as to enable the company to pay out exactly as much as it receives is called the *net annual premium* and is the basis for determining the *gross annual premium* that the company actually charges the insured. Some amount, called the *loading*, is added to the net annual premium to provide for expenses such as officers' salaries, office expenses, agents' commissions, fees to medical examiners, and so forth. The loading depends upon the kind of policy selected and upon other factors.

The following illustrations show how the principle of equivalent obligations is used to calculate the net annual premium of an insurance policy that contains few or many provisions.

Illustrations

1. Frank Ball, whose age to the nearest birthday is 25, wishes to know the net annual premium, p, for an ordinary life policy of \$1000.

According to the assumptions made, each of the l_{25} persons in the group, of which Ball is one, takes such a policy.

The company will collect pl_{25} at the date the policies are issued and pl_{26}, pl_{27}, and so on at the beginning of each of the following years. The value of these collections at the date of birth is

$$p[l_{25}v^{25} + \ .\ .\ .\ + l_{95}v^{95}] = p\sum_{25}^{95} l_xv^x = p\sum_{25}^{95} D_x = pN_{25}.$$

The company will pay, on account of deaths, $1000d_{25}$ at the end of the first year, $1000d_{26}$ at the end of the second year, and so forth. The value of these payments at the date of birth is

$$1000[d_{25}v^{26} + \ .\ .\ .\ + d_{95}v^{96}] = 1000\sum_{25}^{95} d_xv^{x+1} = 1000\sum_{25}^{95} C_x = 1000M_{25}.$$

The equation therefore is

$$pN_{25} = 1000M_{25},$$

and

$$p = 1000\frac{M_{25}}{N_{25}} = 1000 \times \frac{11{,}631}{770{,}114} = 15.10.$$

The net annual premium is \$15.10.

2. Arthur Bird, 30, wishes to know the net annual premium, p, necessary for a policy that contains the provisions:

(a) He is to make 20 equal annual premium payments, p, beginning when the policy is issued.

(b) His beneficiaries are to receive \$1000 when he dies, provided the policy is still in force.

(c) He is to receive 10 annual payments of \$500, beginning at age 60, provided he lives.

(d) He will surrender the policy at age 75 if he is alive then and will receive \$2000.

In the calculation of the net annual premium it is again assumed that each of the l_{30} persons in the group takes a policy having the same provisions, each of which must be considered. The diagram, Fig. 30, showing important dates and the numbers alive at those dates, is helpful in forming the equation.

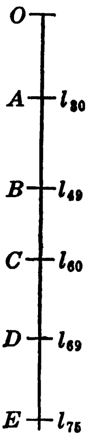


Fig. 30.

O = date of birth.
A = date when the policy is issued.
B = date when Bird makes his last premium payment.
C = date when Bird receives the first \$500.
D = date when Bird receives the last \$500.
E = date when Bird surrenders the policy and receives \$2000.

The value at date O of the several provisions are:

Provision (a) $p\sum_{30}^{49} l_x v^x = p\sum_{30}^{49} D_x = p(N_{30} - N_{50})$.

Provision (b) $1000\sum_{30}^{74} d_x v^{x+1} = 1000\sum_{30}^{74} C_x = 1000(M_{30} - M_{75})$.

Provision (c) $500\sum_{60}^{69} l_x v^x = 500\,(N_{60} - N_{70})$.

Provision (d) $2000 l_{75} v^{75} = 2000\, D_{75}$.

The equation therefore is

$$p(N_{30} - N_{50}) = 1000(M_{30} - M_{75}) + 500(N_{60} - N_{70}) + 2000D_{75},$$

and

$$p = 1000 \times \frac{(M_{30} - M_{75}) + \frac{1}{2}(N_{60} - N_{70}) + 2D_{75}}{N_{30} - N_{50}}$$

$$= 1000 \times \frac{8668 + 27{,}569 + 3976}{415{,}141} = 96.87.$$

The net annual premium is \$96.87.

Note: Provision (c) accounts for $\frac{27{,}569}{40{,}213}$, or 68.557%, of the net premium, and without provision (c) the net premium would be \$30.46.

122. Comparison of premiums. Table XV shows the net annual premium on a \$1000 policy taken at ages 21 to 41 for each of the 4 standard policies. It also shows the actual or gross annual premiums. Hence it is easy to determine the loading in each case.

The tabulation shows that a man of 35 who can afford to pay approximately \$250 a year for insurance may carry:

(a) \$5000 twenty year endowment,
(b) 7000 twenty payment life,
(c) 9000 ordinary life, or
(d) 18,000 ten year term.

Exercise 91

1. Table XV gives the answers to 84 problems that require the net annual premium. Verify at least one entry for each standard policy.

2. Show that the formulas for the net annual premium at age x for the standard policies mentioned are:

(a) ordinary life, $p = 1000\,\dfrac{M_x}{N_x}$.

(b) 20 payment life, $p = 1000\,\dfrac{M_x}{N_x - N_{x+20}}$.

(c) 20 year endowment, $p = 1000\,\dfrac{M_x - M_{x+20} + D_{x+20}}{N_x - N_{x+20}}$.

(d) 10 year term, $p = 1000\,\dfrac{M_x - M_{x+10}}{N_x - N_{x+10}}$.

3. Charles Blake desires a life insurance policy that will provide for a payment of \$5000 to his beneficiaries when he dies and a life annuity of \$1000 to himself beginning at age 60.

Find the net annual premium:

(a) if the policy is taken at age 25 and Blake is to make 30 equal annual payments;

(b) if the policy is taken at age 35 and Blake is to make 20 equal annual payments;

(c) if the policy is taken at age 20 and Blake makes a single payment when the policy is issued;

(d) if Blake makes a single payment at age 40.

4. Abel Field, aged 35, desires a life insurance policy that will provide for a payment of \$10,000 to his beneficiaries when he dies. Show that each of 2 equal net premium payments to be made at ages 35 and 40 to the nearest dollar is \$2054.

5. If, in example 4, Field is to make 3 equal net premium payments at ages 35, 40, 45, find how much each payment should be.

6. If, in example 4, Field is to make 15 annual net premium payments, beginning immediately, of which the first 5 shall be \$300 each, find how large each of the remaining 10 net annual premium payments should be.

7. Henry Carter makes a single premium payment, p, at age 45, for which he is to receive \$500 annually for life beginning at age 46.

$$\text{Show that } p = 500\,\frac{\sum\limits_{46}^{95} l_x v^x}{l_{45}v^{45}} = 500\,\frac{N_{46}}{D_{45}} = \$7543.$$

Note: The symbols N and D were used originally to indicate numerator and denominator—that is, $D_x = l_x v^x$ and $N_x = \sum\limits_{x}^{95} l_x v^x$.

123. Policy reserve. The net premium for each of l_x life insurance policies of \$1000 at age x for 1 year is

$$\frac{1000 d_x v^{x+1}}{l_x v^x} = 1000\,\frac{C_x}{D_x}.$$

But if the policies were renewed year by year, the premium for any year would be higher than for the preceding year, since C_x increases and D_x decreases.

The *level* annual premium for long term insurance is a sort of average. It is more than sufficient to meet the obligations of the company in the early years of the life of the policy. The excess collections constitute a *reserve* fund from which deficiencies are met in the later years. A policy-holder who surrenders his policy at any time is entitled to a share of the reserve. The *cash surrender* value of a policy depends upon the reserve.

To calculate the reserve R on any policy at any date, the reserve on the policies of the entire group is considered and values are compared, as in the calculation of net premium, at the date of birth.

One method of calculating the reserve at any date is to consider the transactions that were made to that date. Another method is to consider the transactions that would be required after that date. The two methods lead to different formulas for the reserve, which are, however, algebraically identical. Since one of the formulas may require less arithmetic labor than the other, the student is advised to become familiar with both methods shown in the following

Illustration

Andrew Bent holds a life insurance policy taken at age 30 that provides:

(a) He is to make 20 equal annual premium payments beginning at the date the policy is issued.

(b) He is to receive \$500 a year for life beginning at age 60.

(c) His beneficiaries are to receive \$1000 when he dies.

The net annual premium is

$$p = \frac{1000M_{30} + 500N_{60}}{N_{30} - N_{50}} = 122.40.$$

The gross annual premium exceeds \$122.40, but the excess is absorbed by expenses so that the reserve depends upon the net and not upon the gross premium. It is required to calculate the reserve R for each of the l_{45} policies outstanding at age 45.

Method I

The value of the premiums collected must balance the sum of the values of the payments that have been made by the company and of the reserve it still holds. Hence

$$p\sum_{30}^{44} l_x v^x = 1000\sum_{30}^{44} d_x v^{x+1} + Rl_{45}v^{45},$$

or $$p(N_{30} - N_{45}) = 1000(M_{30} - M_{45}) + RD_{45},$$

and $$R = \frac{p(N_{30} - N_{45}) - 1000(M_{30} - M_{45})}{D_{45}} = 2468.$$

Method II

The sum of the values of the reserve on hand and of the premiums still to be collected must balance the payments still to be made. Hence

$$Rl_{45}v^{45} + p\sum_{45}^{49} l_xv^x = 1000\sum_{45}^{95} d_xv^{x+1} + 500\sum_{60}^{95} l_xv^x,$$

or $$RD_{45} + p(N_{45} - N_{50}) = 1000M_{45} + 500N_{60},$$

and $$R = \frac{1000M_{45} + 500N_{60} - p(N_{45} - N_{50})}{D_{45}} = 2468.$$

Note. 1. The difference between the values of R found by the two methods is

$$\frac{p(N_{30} - N_{50}) - 1000M_{30} - 500N_{60}}{D_{45}}.$$

But since $$p = \frac{1000M_{30} + 500N_{60}}{N_{30} - N_{50}},$$

this difference is zero.

2. If the reserve were required at age 55, the second method would be preferable, since in that case p would not appear in the formula.

Exercise 92

1. An insurance policy was issued to Henry Parker, at age 30, providing for 20 equal annual premium payments, for Parker to receive \$1000 a year for life beginning at age 65, and for Parker's beneficiaries to receive \$2000 when he died. Find:

(a) the net annual premium;

(b) the reserve when Parker is 45 just before he pays the required premium;

(c) the reserve when Parker is 63;

(d) the reserve when Parker is 70, just before he receives the annual payment of \$1000.

2. Use Table XV, if necessary, to calculate the reserves on the following \$1000 policies taken at age 25:

(a) On a 10 year term policy after it has been in force: (1) 5 years, (2) 10 years.

(b) On an ordinary life policy after it has been in force 20 years.

(c) On a 20 payment life policy after it has been in force: (1) 10 years, (2) 20 years, (3) 30 years.

(d) On a 20 year endowment policy after it has been in force (1) 10 years, (2) 20 years.

3. Three persons, each 25 years old, took, respectively, an ordinary life policy, a 20 payment life policy, and a 20 year endowment policy for \$1000, paying gross annual premiums of \$20.14, \$29.98, \$49.21, the net annual premiums being, respectively, \$15.10, \$22.52, \$39.14.

Each of the three set aside \$49.21 a year to be used partly for insurance and partly for deposit in a bank that allowed interest at $3\frac{1}{2}\%$ per annum effective.

If all three are alive at age 45, surrender their policies and draw the cash from the bank, find the amount each will have, assuming that the surrender value is the same as the reserve.

124. Average life expectation. The mortality table enables us to calculate how many years longer a person at any age, x, may expect to live.

Each of the l_x persons alive at age x will live 1 year except the d_x persons that die during the year. If we assume that, on the average, deaths occur at the middle of the year, the total number of years that the entire group will live during this year is $l_x - \frac{1}{2}d_x$. In the following year, the total is $l_{x+1} - \frac{1}{2}d_{x+1}$, and so on. The total for the entire group, l_x, until the last one dies is

$$\sum_x^{95} l_x - \tfrac{1}{2}\sum_x^{95} d_x = \sum_x^{95} l_x - \tfrac{1}{2}l_x.$$

Hence the average for one of the group is

$$\left[\sum_x^{95} l_x - \tfrac{1}{2}l_x\right] \div l_x = \frac{\sum_x^{95} l_x}{l_x} - \tfrac{1}{2}.$$

Thus the average life expectation at age 30 is $\dfrac{\sum_{30}^{95} l_x}{l_{30}} - \frac{1}{2}$.

Table XIII shows the average life expectation for persons from age 10 to age 95. But we must bear in mind that this mortality table was prepared more than 70 years ago and that at present the average life expectation is higher. One of the mortality tables prepared by the Bureau of the Census, United States Government, shows that, of 81,286 alive at 10 years of age, 330 are alive at age 95 and 1 is alive at age 104.

Note: Different mortality tables are used for the calculation of net premiums for males and for females, the life expectation of females being higher than that of males. Adjustments are usually made because of the extra risk taken by the insurance company upon the life of a person whose occupation is extra hazardous or of a person whose physical condition is below normal, by adding a number of years to the age of the insured. Thus a man of 35 may have to pay the premiums ordinarily required of a man of 40.

Part III

STATISTICAL METHODS

CHAPTER XI

PROBABILITY

125. Equally likely events. Two events are equally likely to occur when there is no reason to expect one to occur rather than the other. A coin when tossed is as likely to fall showing a head as a tail. A card drawn at random from 10 cards numbered from 1 to 10 is as likely to be numbered 7 as 3. A top having 5 equal faces numbered from 1 to 5 and spun until it falls to rest is as likely to come to rest on one face as on another.

Dice are cubes whose 6 faces are marked from 1 to 6 so that the sum of the numbers on two opposite faces is 7. If a die (singular of dice) is cast, one number is as likely to be at the top as another.

A full deck of poker or bridge cards consists of 52 cards divided into 4 suits—hearts and diamonds, both red, and clubs and spades, both black. Each suit has 13 denominations, ace or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and three picture or face cards, jack, queen, king. If a card is drawn at random from a full deck, it is as likely to be a spade as a heart or as likely to be a 10 as an ace.

Of course it is possible to have coins, dice, tops, or cards so made that two different events are not equally likely to occur, but in all subsequent discussions we shall assume that the coins, dice, and other items are not biased.

126. Independent and mutually exclusive events. Two events are independent if the occurrence of one cannot influence the occurrence of the other. If two dice are cast, a 5 showing on one die and a 3 on the other are independent.

Two events are mutually exclusive if the occurrence of

one makes it impossible for the other to occur. If a coin is tossed, a head and a tail showing are mutually exclusive events.

127. Success and failure. In the case of the 5-sided top, we may consider that we score a success if it falls on face 3 and that we score a failure if it falls on any other face. We may also consider that falling on an even-numbered face is a success and falling on an odd-numbered face is a failure. Since every way that the top falls is either a success or a failure, the sum of all the possible successes and failures is 5, the total number of possible occurrences.

128. Probability and odds. The mathematical probability, p, of the future occurrence of an event is defined by

$$p = \frac{s}{s+f} = \frac{s}{n},$$

where s is the number of equally likely ways of succeeding, f is the number of ways of failing, and n is the total number of ways of succeeding and failing. The probability of failure, q, is $q = f/n$.

The odds in favor of the occurrence of an event are s/f and the odds against its occurrence are f/s.

Thus in the case of the top the probability that it will rest on an odd-numbered face is $\frac{3}{5}$ since in this case $s = 3$, $f = 2$, and $n = 5$. $p = \frac{3}{5}$ means that in a great many trials, say one million, in approximately $\frac{3}{5}$ of the total number of trials, or in approximately 600,000 trials, the top will rest on an odd-numbered face. It does not mean that in 10 trials 6 successes may be expected.

The odds in favor of the top coming to rest on an odd number are $\frac{3}{2}$ or 3 to 2, which means that, if A and B play a game with such a top, A winning if it rests on an odd number and losing if it rests on an even number, A should

bet 3 cents to every 2 cents that B bets in order that the players shall have equal chances of winning.

Exercise 93

1. In the case of the 5-sided top, state the probability that it will rest on (a) 3; (b) an even number; (c) an odd number; (d) a prime number; (e) a number which is the square of an integer.

2. A card is drawn at random from a full deck of bridge cards. State the probability that the card is (a) red; (b) a spade; (c) an ace; (d) the 10 of hearts; (e) not an ace; (f) not a club; (g) not a face card.

3. A die is cast. State the probability that the number at the top is (a) 5; (b) even; (c) odd; (d) prime; (e) divisible by 3.

4. A date is selected at random. State the probability that it is (a) a Sunday; (b) a weekday; (c) a day whose initial letter is T.

5. One member of a baseball nine is sick. State the probability that it is (a) the catcher; (b) not the pitcher.

6. State the odds in favor of and the odds against each of the occurrences in examples 1 to 5.

7. A and B play a game with a die, A to win if the number that shows at the top is divisible by 2 and B to win if the number is divisible by 3, and if 6 shows neither wins. Show that A should bet 2 to B's 1.

129. Independent events. A coin can fall in either of 2 ways, head (h) or tail (t). A die can fall in any one of 6 ways. The result of tossing a coin and casting a die may be any one of the 12 occurrences $h1, h2, \ldots, t1, t2, \ldots$. A generalization of this problem is as follows: If one event P can occur in a different, equally likely ways, and another event Q can occur in b different, equally likely ways, then if P and Q are independent, the two events P and Q can occur simultaneously or in succession in ab different ways. For, any one of the a ways of the occurrence of P may be associated with any one of the b ways of the occurrence of Q.

The 5-sided top is spun, and a die is cast. The probability that the top will rest on an even-numbered face is

$p_1 = \frac{2}{5}$, and the probability that the die will rest on a prime-numbered face is $p_2 = \frac{4}{6}$. The probability of both occurrences is $p_1p_2 = \frac{2}{5} \times \frac{4}{6}$, for, counting successes, the top may fall 2 ways and the die 4 ways. The two may fall 2×4 successful ways. Counting totals, the top may fall 5 ways and the die 6 ways. The two may fall 5×6 total ways. Hence the probability of both being successes is

$$\frac{2 \times 4}{5 \times 6} = \frac{2}{5} \times \frac{4}{6}.$$

In general, if the separate probabilities of the occurrences of two independent events are $p_1 = s_1/n_1$ and $p_2 = s_2/n_2$, the probability of the occurrence of both events, either simultaneously or in succession, is

$$p_1p_2 = \frac{s_1s_2}{n_1n_2}.$$

130. Mutually exclusive events. If the probabilities of the occurrences of two mutually exclusive events are $p_1 = s_1/n_1$ and $p_2 = s_2/n_2$, the probability that either of the events will occur is

$$p = p_1 + p_2 = \frac{s_1}{n_1} + \frac{s_2}{n_2}.$$

Since failure and success are mutually exclusive,

$$p + q = \frac{s}{n} + \frac{f}{n} = \frac{s+f}{n} = \frac{n}{n} = 1.$$

The probability 1 indicates certainty, and any other probability is numerically less than 1.

Illustration

A and B play a game with the 5-sided top, A winning if the top rests on an odd number and B winning if it rests on an even number. The probability that A wins in any trial is $\frac{3}{5}$, and that B wins, $\frac{2}{5}$. Suppose the trials are to continue until either has won 4 times. In the first 3

trials, A won once and B twice. Find the probability that A wins the series.

The game may continue to 5, 6, or 7 trials. For A to win the series by the end of the 6th, he must win the 4th, 5th, and 6th, the probability of which is $(\frac{3}{5})^3$. For A to win by the end of the 7th, he must win the 7th and lose only one of the 4th, 5th, and 6th. The probability of losing the 4th and winning the other 3 is $\frac{2}{5} \times (\frac{3}{5})^3$, and the probabilities of losing the 5th only or the 6th only are also $\frac{2}{5}\ (\frac{3}{5})^3$ for each case. Therefore the probability that A wins the series is

$$(\tfrac{3}{5})^3 + 3 \times \tfrac{2}{5}\ (\tfrac{3}{5})^3 = \tfrac{11}{5} \times (\tfrac{3}{5})^3 = \tfrac{297}{625}.$$

The probability that B wins the series is $1 - \frac{297}{625} = \frac{328}{625}$, and the odds in favor of B winning the series are 328 to 297, although the odds in favor of B winning in a single trial are 2 to 3.

Exercise 94

1. A box contains 6 black and 4 white balls. If a ball is drawn at random, find the probability that it is (a) black; (b) white; (c) either black or white.

2. A box contains 6 black, 4 white, and 3 blue balls. If a ball is drawn at random, find the probability that it is (a) black; (b) white; (c) blue; (d) not black; (e) not white; (f) not blue; (g) either black or white; (h) neither black nor white.

3. One of the figures of the number 65,645,654 is written indistinctly. Find the probability that the indistinct figure is (a) 6; (b) 5; (c) 4; (d) not 6; (e) not 5; (f) not 4; (g) 4 or 6; (h) 4, 5, or 6.

4. One hundred cards are marked 1,2,3, . . . , 100. If a card is drawn at random, find the probability that the number on it is (a) even; (b) odd; (c) divisible by 5; (d) divisible by 7; (e) one in which the last figure is 7; (f) one in which the last figure is either 6 or 9.

5. A box contains 6 black and 4 red balls. A ball is drawn at random, and a card is drawn at random from a full deck of bridge cards. Find the probability that (a) both the ball and the card are black; (b) both are red; (c) the ball is black and the card is a picture; (d) the ball is red and the card is marked 2, 3, 4, or 5.

6. A and B play the game with the 5-sided top as explained in the illustration, and in 4 trials A won 3 times and B once. Find the probability that (a) A wins the series; (b) B wins the series.

7. Find the probability that A wins the series if, after 4 trials, (a) A won 3 times; (b) A won twice; (c) A won once.

8. The game is played by A and B with a die, A winning if an odd number shows and B if an even number shows. Find the probability that A wins the series if, after 3 trials, (a) A won once; (b) A won twice; (c) A won three times.

9. The conditions of the game of example 8 are changed so that A wins if the number showing is prime. Find the probabilities for cases (a), (b), (c).

10. A full deck of bridge cards is shuffled, and the bottom card is noted after each shuffle. Show that the probability that in 4 shuffles, cards of the 4 different suits are at the bottom is $\frac{3}{32}$.

11. Four numbers are written at random and are multiplied together. Find the probability that the product is (a) an odd number; (b) an even number.

12. In the game commonly played with a pair of dice, the player wins if his initial throw is 7 or 11 and loses if his initial throw is 2, 3, or 12. If any other total shows, say 8, the player continues to cast the dice until either a total of 8 shows, in which case he wins, or a total of 7 shows, in which case he loses. Show that totals may turn up as follows:

(a) 2 or 12,	1 way each;	(e) 6 or 8,	5 ways each;
(b) 3 or 11,	2 ways each;	(f) 7,	6 ways;
(c) 4 or 10,	3 ways each;	(g) all totals, 36 ways.	
(d) 5 or 9,	4 ways each;		

13. Find the odds in favor of the player to win on the initial throw.

14. Find the odds against the player if his initial throw is 8 and he will win by throwing (a) an 8 before a 7; (b) an 8 or a 6 before a 7; (c) an 8 and a 6 before a 7; (d) a 6 and then an 8 before a 7.

15. Find the odds against the player if his initial throw is 4 and he will win by throwing (a) a 4 before a 7; (b) a 4 or a 10 before a 7; (c) a 4 and a 10 before a 7; (d) a 10 and then a 4 before a 7.

16. Find the odds against the player if his initial throw is 5 and he will win by throwing (a) a 5 before a 7; (b) a 5 or a 9 before a 7; (c) a 5 and a 9 before a 7; (d) a 9 and then a 5 before a 7.

17. Find the odds against the player if his initial throw is 10 and he will win by throwing a 5, then a 6, and then an 8 before throwing a 7.

18. When two dice are cast, explain why the number of ways that the total 9 appears is the same as for the total 5.

19. When 3 dice are cast, (a) find how many different ways the dice can fall; (b) find the largest and the smallest total; (c) show that the number of ways that the total 10 appears is the same as for the total 11.

20. Dominoes are oblong pieces divided into two squares. Each square is either blank or has a number of spots on it like those on dice. Find how many pieces there are if the marks run from double blank to (a) double 6; (b) double 9.

21. Find the probability that the sum of the spots on the two squares is 6 if a domino is picked at random from a set marked from double blank to (a) double 6; (b) double 9.

22. A and B each pick a domino at random, one from a set marked from double blank to double 6 and the other from a set marked from double blank to double 9. Find the probability that the dominoes picked have 6 and 9 spots respectively.

131. Statistical probability. It is not always possible to calculate the number of successes and failures. From a record of past happenings, however, the probability of the occurrence of a similar event is calculated on the assumption that the same natural laws operate. Thus if, in a given locality, precipitation (rain or snow) occurred in 150 days in each of several years, it is assumed that the probability of precipitation for any day is $\frac{150}{365}$.

The mortality table is one of the most reliable of such records of past events, and by means of it probability calculations are made as in the following

Illustration

John is 25, and Bill is 20. The probability that John will be alive 10 years hence is $a = l_{35}/l_{25}$ and the probability that he will be dead 10 years hence is $1 - a = 1 - l_{35}/l_{25}$. Similarly, the probability that Bill will be alive then is $b = l_{30}/l_{20}$ and that Bill will be dead, $1 - b = 1 - l_{30}/l_{20}$.

At the end of 10 years, both may be alive, both may be dead, John may be alive and Bill dead, or John may be dead and Bill alive. The probabilities of these results are

$$p_1 = ab,\ p_2 = (1-a)\ (1-b),\ p_3 = a\ (1-b),\ p_4 = b\ (1-a).$$

The sum of the four probabilities must be 1 since it is a certainty that one of the 4 results must occur, and

$$p_1 + p_2 + p_3 + p_4 = ab + 1-a-b + ab + a-ab + b-ab = 1.$$

Exercise 95

1. Past records show that, during the months of June, July, and August, there were 23 rainy days. Henry will have a vacation of one week in July. Find the probability that during his vacation (a) there will be no rain; (b) it will rain every day; (c) it will rain on one day only.

2. Fred and Joe are now 12 and 18 years old, respectively. Indicate the probability that 9 years hence (a) both will be alive; (b) both will be dead; (c) only one will be alive.

3. Three men, A, B, C, are each 50 years old. Indicate the probability that 20 years hence (a) all three will be alive; (b) all three will be dead; (c) only one will be alive; (d) only one will be dead; (e) at least one will be alive; (f) at least one will be dead.

4. The records of many hospital cases show that the results of a disease X are that 20% of the cases die within a year after treatment is begun. Five cases are admitted for treatment. Find the probability that (a) none will die within a year; (b) only one will die within a year; (c) all will die within a year; (d) only one will survive at the end of a year; (e) at least one will survive at the end of a year.

132. Permutations and combinations. It is often possible to avoid counting the number of ways that one or more events can happen, succeed, or fail, by making a simple calculation. How many different ways a set of 4 can be made from a total of 6 things depends upon (a) whether or not the order in which the 4 are placed makes a difference, and (b) whether or not 2 or more of the 6 are identical or indistinguishable.

If the 6 things are all different and the set of 4 is different when arranged in a different order, each set of 4 is called a *permutation*, and the number of permutations is indicated by ${}_6P_4$.

If the 6 things are all different and the set of 4 is not different when arranged in a different order, each set is a *combination*, and the number of combinations is indicated by ${}_6C_4$.

Thus if the 6 things are the letters a, b, c, d, e, f, and a set of 4 is to make a word (not necessarily pronounceable), two sets such as $b\ a\ d\ e$ and $a\ b\ e\ d$ are different, although both sets consist of the same letters, and the number of different words of 4 letters each is ${}_6P_4$. But if the letters represent algebraic numbers, so that $b\ a\ d\ e$ means $b \times a \times d \times e$, the product $b\ a\ d\ e$ is not different from the product $a\ b\ e\ d$, and the number of different products is ${}_6C_4$.

Exercise 96

Indicate by the proper symbols the answers to the following:

1. The number of different lines that are determined by 6 points if no 3 points are collinear (on a line).

2. The number of different triangles that are determined by (a) 6 points, if no 3 are collinear, and no 2 lines determined by the points are parallel; (b) 6 lines, if no 3 are concurrent (pass through the same point), and no 2 are parallel.

3. The number of different committees of 5 that can be selected from 8 persons.

4. The number of different committees of 3 that can be selected from 8 persons.

5. Show that the answers to examples 3 and 4 are the same and, in general, that ${}_nC_r = {}_nC_{n-r}$.

6. A class consists of 10 boys and 6 girls. How many ways can a committee of 4 boys and 2 girls be made?

7. In example 6, John is one of the boys and May is one of the girls. Indicate the number of ways of making the committee if (a) John is on the committee; (b) May is on the committee; (c) John is on the committee and May is not; (d) May is on the committee and John is not; (e) John and May are both on the committee; (f) Neither John nor May is on the committee.

8. Indicate how many different numbers of 3 figures each can be made by using the figures 1, 2, 3, 4, 5, 6, if (a) no figure is to be used more than once in any number; (b) a figure may be used once, twice, or three times in any number; (c) the numbers are to be odd under condition (a); (d) the numbers are to be even under condition (a); (e) the numbers are to be odd under condition (b); (f) the numbers are to be even under condition (b); (g) the middle figure is to be 5 under condition (a); (h) the middle figure is to be 5 under condition (b).

133. Formula for $_nP_r$. To find the number of permutations that can be made from n different things if each set is to consist of r things, consider r successive places to be filled, Fig. 31. Since any one of the n things can go into the first place, this place can be filled n ways. For the second place there are only $n - 1$ things from which to choose, and this place can be filled $n - 1$ ways. The acts of filling the 2 places being independent, the number of ways of filling the 2 places is $n\ (n - 1)$ or $_nP_2 = n\ (n - 1)$.

1	2	3	4
n	n-1	n-2	n-3

Fig. 31.

Similarly, the first 3 places can be filled $n\ (n - 1)\ (n - 2)$ ways, or $_nP_3 = n(n - 1)\ (n - 2)$. Hence

$$_nP_1 = n,$$
$$_nP_2 = n(n-1),$$
$$_nP_3 = n(n-1)\ (n-2),$$
$$_nP_4 = n(n-1)\ (n-2)\ (n-3),$$
$$\cdots\cdots$$
$$_nP_{10} = n(n-1)\ (n-2)\ .\ .\ .\ (n-9),$$
$$_nP_r = n(n-1)\ (n-2)\ .\ .\ .\ (n-r+1).$$

If $r = n$,

$$_nP_r = {}_nP_n = n(n-1)\ (n-2)\ .\ .\ .\ 3 \cdot 2 \cdot 1.$$

The product for $_nP_n$ is indicated by writing $n!$, read "n factorial."

A product such as $8 \cdot 7 \cdot 6$ may be written

$$\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8!}{5!}$$

Hence we may write

$${}_8P_3 = 8 \cdot 7 \cdot 6 = \frac{8!}{5!} = \frac{8!}{(8-3)!},$$

and, in general,

$${}_nP_r = \frac{n!}{(n-r)!}.$$

If $r = n$,

$${}_nP_n = \frac{n!}{0!}.$$

But since ${}_nP_n = n!$, the form $0!$, if it is to have a meaning, must equal 1.

To calculate with factorial numbers that are moderately large, it is convenient to use a table for $\log (n!)$, Table XVII, which is made from the table of common logarithms by successive additions. Thus

$$\log (3!) = \log (3 \cdot 2 \cdot 1) = \log 3 + \log 2 + \log 1,$$

and

$$\log (4!) = \log (4 \cdot 3!) = \log 4 + \log (3!),$$

and so on.

Exercise 97

1. Find the value of: (a) ${}_5P_2$; (b) ${}_6P_3$; (c) ${}_5P_5$; (d) ${}_{10}P_3$; (e) ${}_9P_3$; (f) ${}_8P_3$.

2. Calculate the value of:

(a) $\frac{5!}{3!}$ (d) $\frac{9!}{6!}$

(b) $\frac{6!}{3!}$ (e) $\frac{8!}{7!}$

(c) $\frac{10!}{8!}$ (f) $\frac{10!}{6!4!}$

3. Show that:

(a) $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 = 7!2^7$; (b) $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 = \frac{13!}{6!2^6}$; (c) $7!2^7 \cdot \frac{13!}{6!2^6} = 14!$

4. Express in terms of factorial numbers:

(a) $52 \cdot 51 \ldots 46 \cdot 45$; (b) $20 \cdot 19 \ldots 12 \cdot 11$.

5. Find an approximate value of ${}_{52}P_{39}$ by using the table of the logarithms of factorial numbers, Table XVII.

6. Show from the table log $(n!)$ that:

(a) $\log 11 = 1.041393$ (b) $\log 84 = 1.924279$.

134. Formula for ${}_nC_r$. The number of combinations of 4 that can be made from 10 different things is ${}_{10}C_4$. In combinations, the order of the elements in the set is disregarded. If the order were considered, each set of 4 could be made into 4! new sets and there would be a total of $4!\,{}_{10}C_4$ new sets. But the result would now be the same as if we had selected our sets of 4 by making permutations of which there would be ${}_{10}P_4$. Hence

$$4!\,{}_{10}C_4 = {}_{10}P_4,$$

and

$${}_{10}C_4 = \frac{{}_{10}P_4}{4!} = \frac{10!}{6!4!},$$

and, in general,

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}.$$

Every time a set of 4 is picked from among 10, there is left a set of 6. Hence there are as many combinations of 4 that can be made from 10 as there are combinations of 6, or ${}_{10}C_4 = {}_{10}C_6$, and, in general, ${}_nC_r = {}_nC_{(n-r)}$.

Illustrations

1. How many words can be made from the 7 consonants *b c d f g h k* and the 5 vowels *a e i o u* if each word is to contain 4 different consonants and 2 different vowels?

A set of 4 consonants, without regard to order, can be selected in ${}_7C_4 = {}_7C_3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35$ ways.

A set of 2 vowels can be selected in ${}_5C_2 = \frac{5 \cdot 4}{1 \cdot 2} = 10$ ways.

Since the selection of consonants and the selection of vowels are independent, a set of 4 consonants and 2 vowels can be selected in $35 \times 10 = 350$ ways.

The 6 letters of each set can be arranged in ${}_6P_6 = 6! = 720$ ways. Therefore the number of words is ${}_7C_4 \times {}_5C_2 \times {}_6P_6 = 35 \times 10 \times 720 = 252{,}000$.

2. Find the probability that a word selected at random from among the 252,000 words of illustration 1 contains the letters *d* and *a*.

The letter *d* in the word requires that 3 other consonants from among 6 consonants remaining must have been used. This is possible ${}_6C_3$ ways. The letter *a* in the word requires that 1 other vowel from among the 4 vowels remaining must have been used. This is possible ${}_4C_1$ ways.

The number of possible words that contain the letters *d* and *a* is ${}_6C_3 \cdot {}_4C_1 \cdot {}_6P_6$, and the probability is

$$p = \frac{{}_6C_3 \cdot {}_4C_1 \cdot {}_6P_6}{{}_7C_4 \cdot {}_5C_2 \cdot {}_6P_6} = \frac{8}{35}.$$

3. The probability that neither *d* nor *a* is in the word selected is

$$\frac{{}_6C_4 \cdot {}_4C_2 \cdot {}_6P_6}{{}_7C_4 \cdot {}_5C_2 \cdot {}_6P_6} = \frac{9}{35}.$$

4. The probability that *d* is and *a* is not in the word selected is

$$\frac{{}_6C_3 \cdot {}_4C_2 \cdot {}_6P_6}{{}_7C_4 \cdot {}_5C_2 \cdot {}_6P_6} = \frac{12}{35}.$$

5. The probability that *d* is not and *a* is in the word selected is

$$\frac{{}_6C_4 \cdot {}_4C_1 \cdot {}_6P_6}{{}_7C_4 \cdot {}_5C_2 \cdot {}_6P_6} = \frac{6}{35}.$$

Check: $\frac{8}{35} + \frac{9}{35} + \frac{12}{35} + \frac{6}{35} = 1.$

Exercise 98

Complete the calculations of the solutions to the examples of Exercise 96, page 219.

135. Identical elements. If the n things from which permutations or combinations are to be made are not all different, the problem is more difficult. For such cases we shall consider only the formula for ${}_nP_n$.

How many 6-figure numbers can be made by using all the figures of the number 5 5 3 5 3 5?

There are now 6 elements, of which 4 are identical, 5's, and 2 others are also identical, 3's. If the 5's were distinguishable, say 5_1, 5_2, 5_3, 5_4, and the 3's were also distinguishable, 6! numbers could be made. But four different 5's can be arranged in 4! different orders and two different 3's in 2! different orders. If x different numbers can be made by using all the figures of the number 5 5 3 5 3 5, then if the 5's and the 3's were distinguishable there would be x 4! 2! different numbers. But since the total would then also be 6!,

$$x\,4!\,2! = 6!,$$

and

$$x = \frac{6!}{4!2!} = 15.$$

In general, if of n elements k are identical, l others and m others also identical, ${}_nP_n$ is not $n!$ but

$$\frac{n!}{k!\,l!\,m!}$$

If the question were, "How many 4-figure numbers can be made by using figures of the number 5 5 3 5 3 5?" it would be necessary to enumerate the different ways that 4 figures may be selected or the different ways that 2 figures may be omitted. We may omit

(a) two 5's and form $\frac{4!}{2!2!} = 6$ numbers,

(b) two 3's and form $\frac{4!}{4!} = 1$ number,

(c) a 5 and a 3 and form $\frac{4!}{3!} = 4$ numbers.

Hence $6 + 1 + 4$, or eleven 4-figure numbers, can be made. The student may write the fifteen 6-figure numbers and also the eleven 4-figure numbers.

Such problems also arise when we wish to know how many ways a total of 10 can be made when casting 4 dice. Disregarding the order of the dice and arranging the numbers in decreasing order, we may have:

(1)	6	2	1	1	(5)	4	3	2	1
(2)	5	3	1	1	(6)	4	2	2	2
(3)	5	2	2	1	(7)	3	3	3	1
(4)	4	4	1	1	(8)	3	3	2	2

Now taking the order into account, (1), (2), and (3) may each occur $\frac{4!}{2!}$ or 12 ways; (4) and (8) in 6 ways each; (5) in 24 ways; (6) and (7) in 4 ways each. The total number of ways of making a total of 10 is $3 \times 12 + 2 \times 6 + 24 + 2 \times 4 = 80$.

The number of ways of making a total of 18 is also 80. Since 4 dice can fall 6^4, or 1296, ways, the probability that the sum is 10 is $\frac{80}{1296}$.

Exercise 99

1. A box contains 6 black and 5 white balls, and 4 balls are drawn at random. Find the probability that (a) all 4 are black; (b) all 4 are white; (c) 3 are black and 1 is white; (d) 3 are white and 1 is black; (e) 2 are white and 2 are black.

2. Find how many different words can be made from the letters of the word *p a r a l l e l* if they are to be (a) 8-letter words; (b) 6-letter words.

3. Find how many numbers can be made by using only the figures of the number 334345 if the numbers made are to be (a) 6-figure numbers; (b) 5-figure numbers; (c) numbers between 500 and 5000.

4. Given the points A (2, 3), B (7, 5), C (11, 8) plotted on square ruled paper. You are to move from one point to another in steps of unit spaces by going toward the right and upward, but never obliquely. Find how many ways you can go (a) from A to B; (b) from B to C; (c) from A to C.

5. In example 4, show that the various ways of going from A to B are not equally likely.

6. Seven points lie on the circumference of a circle.

(a) Show that 21 different lines are determined by the 7 points.

(b) If no 2 of the 21 lines are parallel and no 3 are concurrent except those that pass through the 7 given points, show that the number of new points determined by the intersections of the 21 lines is 105.

(c) Show that the number of triangles determined by the 21 lines is 1190.

7. Five parallel horizontal lines are crossed by 8 parallel vertical lines. Show that the number of rectangles formed is 280.

8. Five dice are cast. Show that the frequencies with which the sums from 5 to 30 can appear are:

Sum	*Frequency*	*Sum*	*Frequency*
5 or 30	1	12 or 23	305
6 or 29	5	13 or 22	420
7 or 28	15	14 or 21	540
8 or 27	35	15 or 20	651
9 or 26	70	16 or 19	735
10 or 25	126	17 or 18	780
11 or 24	205		

Note: The total of the frequencies for all the sums from 5 to 30 is $7776 = 6^5$.

9. In a bridge game the 52 cards are dealt to 4 players, 13 cards to each player. Find the probability that one of the players holds (a) 13 cards of one suit; (b) 7 cards of one suit and 6 of another suit; (c) 4 aces, and 9 other cards; (d) 4 aces, 4 kings, and 5 other cards; (e) no aces and no picture cards.

10. When 5 dice are cast, the same number may appear on 2, 3, 4, or 5 dice. Show that the following mutually exclusive sets appear with the frequencies indicated:

(a) 5 numbers are the same, as 44444, 6 ways;

(b) 4 numbers are the same and the fifth is a different number, as 33336, 150 ways;

(c) 3 numbers are the same and the remaining 2 numbers are also the same, as 55544, 300 ways;

(d) 3 numbers are the same and the remaining 2 numbers are different, as 11145, 1200 ways;

(e) 2 numbers are the same, 2 other numbers are also the same, and the remaining number is different from either pair, as 11556, 1800 ways;

(f) 2 numbers are the same and the remaining 3 numbers are different, as 33156, 3600 ways;

(g) No 2 numbers are the same, as 13456, 720 ways.

Check the correctness of the results of (a) to (g) by noting that 5 dice can fall in $6^5 = 7776$ different possible ways.

11. Fifteen men are candidates for a baseball nine. Only A, B, C can pitch, and they will not play any other position. Only D, E, F, G can catch, and they will not play any other position. The remaining men numbered 1, 2, 3, . . . , 8 are competent to play any of the other 7 positions.

(a) Show that the 9 men can be selected in 96 ways.

(b) Show that the probability of A and D being among those selected is $\frac{1}{12}$.

(c) Show that the probability of C and 5 being among those selected is $\frac{7}{24}$.

(d) Show that the probability of B, F, and 8 being among those not selected is $\frac{1}{16}$.

136. Probability and the binomial expansion. In a group of ten events (10 dates, 10 dice cast at one time or 1 die cast 10 times, 10 coins tossed, and so forth), 3 successes and 7 failures can occur in

$$\frac{10!}{3!7!} = {}_{10}C_3 = {}_{10}C_7$$

ways. If the constant probability of success is p and that of failure q, the probability of the occurrence of one of the ways is $p^3\, q^7$, and the probability of 3 successes and 7 failures is ${}_{10}C_3\, p^3\, q^7$. This is one term in the expansion

$$(q + p)^{10} = q^{10} + \frac{10}{1} q^9 p + \frac{10 \cdot 9}{1 \cdot 2} q^8 p^2 + \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} q^7 p^3 + \cdots$$

$$= q^{10} + {}_{10}C_1 q^9 p^1 + {}_{10}C_2 q^8 p^2 + {}_{10}C_3 q^7 p^3 + \cdots .$$

The terms in the expansion of $(q + p)^{10}$, therefore, show respectively the probabilities of 0, 1, 2, 3, . . . , 10 successes and 10, 9, 8, 7, . . . , 0 failures.

In 1000 groups of 10 events, the probable frequencies of 0, 1, 2, . . . , 10 successes to be expected are the terms in the expansion of $1000 (q + p)^{10}$.

Illustrations

1. Six dice are cast. The probability P that exactly 2 aces show is that term in the expansion of $(q + p)^6$ where $p = \frac{1}{6}$, $q = \frac{5}{6}$, in which the exponent of p is 2. That is,

$$P = {}_6C_2 q^4 p^2 = \frac{6 \cdot 5}{1 \cdot 2}\left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right)^2 = \frac{1}{5} \text{ approximately.}$$

In 200 casts of 6 dice we may expect sets of 2 aces and 4 numbers other than aces to appear approximately $200 \times \frac{1}{5}$, or 40, times.

2. The records of a large number of hospital cases show that 20% of those treated for disease K died within a short time. In a group of 10 such cases, the probabilities of 0, 1, 2, . . . , 10 deaths to be expected are the terms in the expansion of $(\frac{4}{5} + \frac{1}{5})^{10}$. They are: 0 deaths, $(\frac{4}{5})^{10} = .1074$; 1 death, ${}_{10}C_1 (\frac{4}{5})^9 (\frac{1}{5}) = .2685$; 2 deaths, ${}_{10}C_2 (\frac{4}{5})^8 (\frac{1}{5})^2 = .3021$; 3 deaths, ${}_{10}C_3 (\frac{4}{5})^7 (\frac{1}{5})^3 = .2014$; and so forth. The probability that not more than 2 deaths will occur is $.1074 + .2685 + .3021 = .6780$; the probability that more than 2 deaths will occur is $1 - .6780 = .3220$.

3. According to the mortality table, $l_{40} = 78{,}106$ and $d_{40} = 765$. The probability, q, that a man of 40 will die before reaching the age of 41 is $q = \frac{765}{78106} = .0098$, and the probability, p, that he will live to be 41 is $p = 1 - .0098 = .9902$.

In a group of 100 men 40 years old, the probability that 2 will die before reaching age 41 is

$${}_{100}C_2 (.0098)^2 (.9902)^{98} = \frac{100 \cdot 99}{1 \cdot 2} (.0098)^2 (.9902)^{98}.$$

Exercise 100

1. Ten coins are tossed. Find the probability that the number of heads showing is (a) 10; (b) 6; (c) 5; (d) 2; (e) at least 4; (f) at the most 4.

2. Five dice are cast. Find the probability that the number of 6's showing is (a) 2; (b) 3; (c) 5; (d) at least 2; (e) at the most 3.

3. Eight dates are written at random. Find the probability that (a) 2 are Sundays and 6 are not Sundays; (b) at least 2 are Sundays; (c) none is Sunday.

4. A box contains 6 black and 4 white balls. A ball is drawn at random, its color noted, and is returned to the box. In 10 such drawings, find the probability that there were drawn (a) 5 white and 5 black balls; (b) 4 black and 6 white balls; (c) 6 black and 4 white balls.

5. A card is drawn from a full deck of bridge cards, its suit and denomination are noted, and the card is returned to the deck. In 5 cards so drawn, find the probability that there were (a) only 1 spade; (b) 1 spade and 1 heart; (c) 5 spades; (d) 1 each of 3 suits and 2 of the 4th suit; (e) 5 black cards; (f) 1 ace; (g) 2 kings.

6. In a particular course of study, 90% of the students attain acceptable grades. Find the probability that in a class of 30 (a) exactly 3 students fail; (b) not more than 3 students fail; (c) none fails.

7. In the expansion of $(p + q)^n$, set $p = q = \frac{1}{2}$ and show that ${}_nC_1 + {}_nC_2 + \ . \ . \ . \ + {}_nC_n = 2^n - 1$.

8. State in words the meaning of the final equation in example 7.

9. Derive the equation of example 7 without using the binomial expansion.

10. There are 6 coins, 1¢, 5¢, 10¢, 25¢, 50¢, 100¢. Show that the number of different sums that can be made by taking 1, 2, 3, . . . , 6 coins at a time is $2^6 - 1 = 63$.

Hint: Each coin may be treated in either of 2 equally likely ways, it may be either taken or not taken.

11. Show that $(p + q)^n = \sum_{x=0}^{x=n} \frac{n!}{x!(n-x)!} p^x q^{n-x}$.

CHAPTER XII

NORMAL PROBABILITY CURVE

137. Tabulation. Statistical data consist of thousands of observations condensed to a relatively small number of items for easier study. The culmination of the study is, if possible, a mathematical equation which shows approximately how the items are related.

For example, a set of 1000 men whose heights vary from about 5 ft. to about 6 ft. are grouped into classes at intervals of 1 inch, say 60, 61, 62, and so on, inches high. Those included in the 62-inch class vary from $61\frac{1}{2}$ to $62\frac{1}{2}$ inches in height, and the middle of the class interval is called the *class mark*. The lower and the upper limits of each class interval are called the *class limits*. If there are several at a class limit, say at $65\frac{1}{2}$ inches, half of them are put into the 65-inch class and half into the 66-inch class, an odd one going into the 66-inch class. The result is a frequency table such as

x (height)	60	61	62	63	64	65	66	67	68	69	70	71
f (frequency)	3	5	10	50	107	170	180	190	165	85	30	5

It is now assumed that the 50 listed as 63 inches are all 63 inches high although they really vary by fractions of an inch from $62\frac{1}{2}$ to $63\frac{1}{2}$. The gradation could be made at class intervals of $\frac{1}{2}$ or $\frac{1}{8}$ of an inch or a smaller fraction of an inch, so that we could have a practically continuous gradation. The same set could also be grouped into classes with respect to weight, and the tabulation would be different but could still be made continuous.

In other cases continuous gradation is not possible. Thus the number of children in a family, or the number of

aces showing when 10 dice are cast, must be integers. Some tabulations are essentially continuous and others are discrete, but all grouped data are called *frequency distributions.*

138. Frequency graphs. The graphic representation of a frequency table consists of a set of points whose abscissas are x and whose ordinates are f. The points are joined in succession by straight lines, and the resulting figure is a *frequency polygon.* Another graphic form, called a *histogram,* consists of a sequence of rectangles symmetrical about the ordinates at the class marks and a class interval in width. If the class interval is a unit, the area of each rectangle is then represented by the same number as its height f, and the sum of the areas of all the rectangles is the same number as the sum of the frequencies. Finally, a smooth curve drawn through the points plotted is called a *frequency curve.*

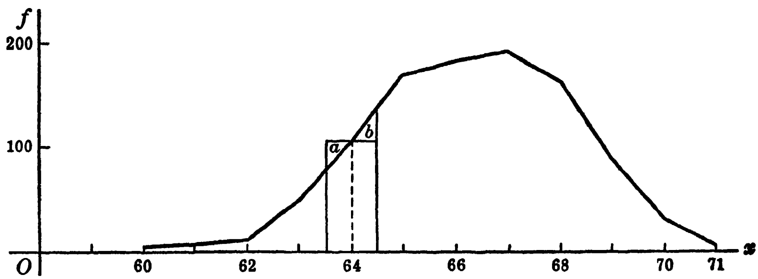


Fig. 32.

The diagram, Fig. 32, shows the frequency polygon and also one of the rectangles of the histogram. The area under the polygon from $x = 63.5$ to $x = 64.5$ exceeds that of the rectangle symmetrical about the ordinate at $x = 64$ by triangle b and falls short of it by triangle a. Since the areas of these triangles are approximately equal, the area under the polygon from 63.5 to 64.5 is approximately equal to the area of the rectangle and therefore to the fre-

quency at $x = 64$. The area under the polygon is approximately equal to the sum of the frequencies. If the frequency curve were drawn, the area under it would be approximately equal to the area under the polygon and therefore to the sum of the frequencies.

If instead of 1000 men, we classified as to height 4000 men at intervals of $\frac{1}{4}$ of an inch, or 10,000 men at intervals of $\frac{1}{10}$ of an inch, the shape of the frequency polygon would be more and more like that of the frequency curve. That is, the frequency curve smoothes out the irregularities in the polygon due to the relatively small sample.

A continuous curve may be represented by an algebraic equation which can be studied mathematically.

If a man is selected from the group at random, the probability that he is in the 64-inch class is $\frac{107}{1000}$ according to the given table. The same probability is found from the histogram, and approximately the same probability is found from the frequency polygon or from the frequency curve.

139. Binomial distribution. The example we have been discussing is a type of frequency distribution often found in actual experience. Such frequency distributions are of the same general pattern as the theoretical distributions of frequencies given by the terms of the expansion of $N\,(q+p)^n$, called *binomial distributions*, which were discussed in the chapter on probability, a sample term being

$$N\,{}_nC_x\,p^x\,q^{n-x} = N\frac{n!}{x!\,(n-x)!}\,p^x\,q^{n-x}.$$

Illustrations

1. A set of 10 coins is tossed 100 times. The probability, p, of a head showing on any coin is $\frac{1}{2}$, and the probability, q, of a tail showing is also $\frac{1}{2}$. The frequencies, f, with which $x = 0, 1, 2, \ldots, 10$ heads may be expected to show in 100 tosses are the terms of the expansion of 100 $(\frac{1}{2}+\frac{1}{2})^{10}$. The values of f to the nearest integer are

x	0	1	2	3	4	5	6	7	8	9	10
f	0	1	4	12	21	24	21	12	4	1	0

2. In a large number of hospital cases, the result of medical treatment by a particular method was that 25% of the patients died within a year. For 100 sets of 10 cases each, the frequencies, f, for $x = 0, 1, 2, \ldots, 10$ deaths, found from $100\left(\frac{3}{4} + \frac{1}{4}\right)^{10}$, are

x	0	1	2	3	4	5	6	7
f	6	19	28	24	15	6	2	0

3. In example 2, if x is the number of patients that survive, the table found from $100\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$ is

x	3	4	5	6	7	8	9	10
f	0	2	6	15	24	28	19	6

The frequencies are proportional to the probabilities in each case. It is evident that the calculation of the probability or of the frequency becomes more laborious when n is large. Thus the probability of exactly 100 deaths in 400 cases treated is

$$ {}_{400}C_{100}\left(\tfrac{1}{4}\right)^{100}\left(\tfrac{3}{4}\right)^{300} = \frac{400!}{100!\,300!} \times \frac{3^{300}}{4^{400}}. $$ (see p. 312)

It is therefore desirable that the binomial distribution be reduced to a mathematical form from which it is not too difficult to make calculations.

140. Probability curve. The frequency curves of the binomial distributions of examples 1, 2, and 3 are shown in Fig. 33. Because these distributions are closely related to probabilities, the curves are called *probability curves*.

Curve A, representing example 1, in which $p = q = \frac{1}{2}$, is symmetrical. The ordinates of this curve would also represent the frequencies with which tails may be expected.

Curve B, representing example 2, in which $p = \frac{1}{4}$, $q = \frac{3}{4}$, is nonsymmetrical or *skew* and has a long tail toward the right.

Curve C, representing example 3, in which $p = \frac{3}{4}$, $q = \frac{1}{4}$, has a long tail toward the left.

The binomial distribution of frequencies arising from $N(q + p)^n$ always gives curves like A, B, or C according as $p = q$, $p < q$, or $p > q$.

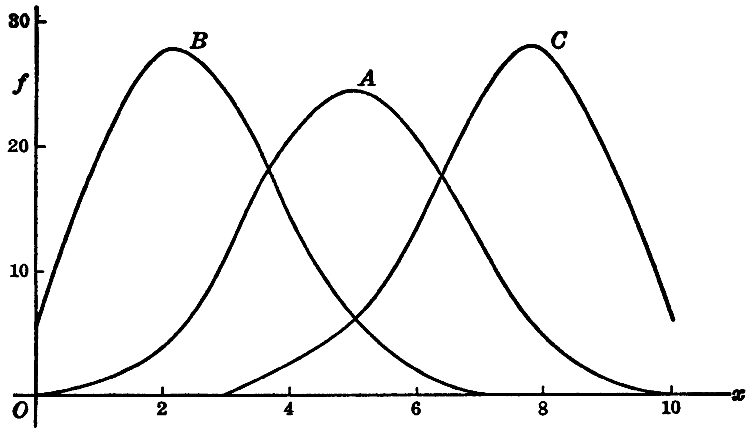


Fig. 33.

We shall see, however, that as n is made larger and larger, the curves B and C tend toward the symmetrical form A regardless of the values that p and q may have.

141. Operations with Σ. We had occasion to use the summation sign Σ (page 196), but, since it will be necessary to use it frequently, the following simple relations are to be carefully noted:

$$(1) \qquad \sum_{i=1}^{i=n} x_i$$

means the sum of all x's having the subscripts $i = 1, 2, 3, \ldots, n$. If no ambiguity can arise, the subscript i and the limits of i are not written, and

$$\Sigma x = x_1 + x_2 + x_3 + \ldots + x_n.$$

$$(2)\ \Sigma 5x = 5x_1 + 5x_2 + \ldots + 5x_n = 5(x_1 + x_2 + \ldots + x_n) = 5\Sigma x.$$

That is, a constant factor may be taken from after Σ and placed before it, or vice versa.

(3) If each of the x's in $\Sigma 5x$ is equal to 1, $\Sigma 5x$ takes the form $\Sigma 5$. But then

$$\Sigma 5x = 5(x_1 + x_2 + \ldots + x_n) = 5(1 + 1 + 1 + \ldots + 1) = 5n.$$

That is, if a constant term appears after Σ, the value of the expression is n times the constant.

$$\begin{aligned}(4)\quad \Sigma(x + y) &= (x_1 + y_1) + (x_2 + y_2) + \ldots + (x_n + y_n)\\ &= (x_1 + x_2 + \ldots + x_n) + (y_1 + y_2 + \ldots + y_n)\\ &= \Sigma x + \Sigma y.\end{aligned}$$

That is, if there are a number of algebraic terms after Σ, the summation is applied to each term separately.

$$\begin{aligned}(5)\ \Sigma(x + y)^2 &= (x_1 + y_1)^2 + (x_2 + y_2)^2 + \ldots + (x_n + y_n)^2\\ &= (x_1^2 + 2x_1y_1 + y_1^2) + (x_2^2 + 2x_2y_2 + y_2^2) + \ldots\\ &\quad + (x_n^2 + 2x_ny_n + y_n^2) = (x_1^2 + x_2^2 + \ldots + x_n^2)\\ &\quad + 2\,(x_1y_1 + x_2y_2 + \ldots + x_ny_n) + (x_1^2 + y_2^2 + \ldots + y_n^2)\\ \Sigma(x + y)^2 &= \Sigma x^2 + 2\Sigma xy + \Sigma y^2.\end{aligned}$$

That is, if indicated operations appear after Σ, the operations are performed in the usual algebraic manner, and Σ is applied to each term of the result.

$$\begin{aligned}(6)\quad \Sigma xy &= x_1y_1 + x_2y_2 + \ldots + x_ny_n;\\ x\Sigma y &= x(y_1 + y_2 + \ldots + y_n);\\ y\Sigma x &= y(x_1 + x_2 + \ldots + x_n);\\ \Sigma x\Sigma y &= (x_1 + x_2 + \ldots + x_n)\,(y_1 + y_2 + \ldots + y_n).\end{aligned}$$

That is, Σxy, $x\Sigma y$, $y\Sigma x$, $\Sigma x\Sigma y$ have different values and must not be confused.

Exercise 101

In the following, x and y mean x_i and y_i, and the limits of summation are from $i = 1$ to $i = n$. Expand and simplify each.

1. $\Sigma(3x - 2)$
2. $\Sigma(3x - 2)^2$
3. $\Sigma(x + y)^2 - \Sigma(x - y)^2$
4. $\Sigma(x + y)^2 - 2\Sigma xy$
5. $\Sigma(x + y)^2 - \Sigma x^2 - \Sigma y^2$
6. $\Sigma(2x + 3y - 4)$
7. $\Sigma x(x - 1)$
8. $\Sigma(x + 2)(x - 3)$
9. $\Sigma(x + y + 1)^2$
10. $\Sigma(2x + 3y - 1)^2$

142. The mean. What is the average height of the 1000 men discussed at the beginning of this chapter? The term *average* has many meanings, but we shall consider only one, the usual meaning, called the *arithmetic mean.* The arithmetic mean, hereafter referred to merely as the mean, is a hypothetical height such that, if each of the 1000 men were of this height, the total height of all would be exactly the same as it is for the actual heights.

The arithmetic mean is indicated by $\bar{x}$ (ex bar). According to the definition,

$$1000\bar{x} = 60 + 60 + 60 + 61 + 61 + 61 + 61 + 61 + \ldots$$
$$= 3 \times 60 + 5 \times 61 + 10 \times 62 + \ldots .$$

In general,

$$\bar{x}\Sigma f = \Sigma fx$$

and

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}.$$

The *deviations* from the mean are the differences $x_1 - \bar{x}$, $x_2 - \bar{x}$, $x_3 - \bar{x}$, and so on, which may be $+$, $-$, or 0. The algebraic sum of the deviations from the mean is

$$\Sigma f(x - \bar{x}) = \Sigma fx - \bar{x}\Sigma f = 0.$$

Instead of calculating the products 3×60, 5×61, and so forth, the value of $\bar{x}$ is found more easily as follows:

Let x' be a number somewhere near the mean, say 67, the value of x corresponding to the greatest frequency, and write $\bar{x} = x' + d$, where d is $+$ or $-$, and $x' = \bar{x} - \text{d}$.

Then

$$\Sigma f(x - x') = \Sigma f[x - (\bar{x} - d)] = \Sigma f(x - \bar{x} + d)$$
$$= \Sigma f(x - \bar{x}) + \Sigma fd = 0 + d\Sigma f.$$

The tabulation is now made, taking $x' = 67$.

x	f	$x - x'$	$f(x - x')$
60	3	−7	− 21
61	5	−6	− 30
62	10	−5	− 50
63	50	−4	−200
64	107	−3	−321
65	170	−2	−340
66	180	−1	−180
→67	190	0	0
68	165	1	165
69	85	2	170
70	30	3	90
71	5	4	20
	1000		−697

Hence $\Sigma f(x - x') = \Sigma fd = d\Sigma f = 1000\ d = -697$, $d = -.697$, and $\bar{x} = x' + d = 67 + (-.697) = 66.303$. That is, the average height is 66.303 inches. Probably there is not a single individual in the group whose height is exactly 66.303 inches, but, if the height of each individual were 66.303 inches, the total height of all would be 66,303 inches, which is the value of Σfx.

Exercise 102

1. Show that Σfx for the tabulation is 66,303.

2. Find $\bar{x} = 66.303$ by taking $x' = 66$ and also $x' = 65$.

3. Find $\bar{x}$ if the weights x in lbs. of 10 boys are

$$x = 70, 72, 73, 75, 78, 80, 83, 86, 90, 93.$$

4. Find $\bar{x}$ if the weights x in lbs. of 90 boys are distributed as follows:

x	70	71	72	73	74	75
f	5	12	18	30	20	5.

5. Find $\bar{x}$ for the table

x	2	3	4	5	6	7	8	9	10	11	12
f	1	2	3	4	5	6	5	4	3	2	1

6. Draw the histogram for the data of example 4.

7. Draw the histogram for the data of example 5.

143. The mean for a binomial distribution. The frequency table of the binomial distribution $N(q + p)^n$ for $x = 0, 1, 2, \ldots, n$ successes are terms like

$$N \frac{n!}{x!(n-x)!} p^x q^{n-x} \quad \text{(see page 229).}$$

$$\Sigma f = N\Sigma\, {}_nC_x\, p^x q^{n-x} = N(q + p)^n = N, \text{ since } q + p = 1.$$

$$\Sigma fx = N\Sigma \frac{n!}{x!(n-x)!} p^x q^{n-x}\, x = Nnp\Sigma \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}.$$

But the value of the sigma factor is $(q + p)^{n-1} = 1$. Therefore

$$\Sigma fx = Nnp$$

and

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{Nnp}{N} = np.$$

That is, the value of $\bar{x}$ in the case of the binomial distribution $N(q + p)^n$ may be found by inspection. It is merely $\bar{x} = np$, which is independent of N. Thus if a set of 60 dice is cast 1000 times and $x = 0, 1, 2, \ldots, 60$ aces, then, since the probability of an ace showing on a die is $p = \frac{1}{6}$, $\bar{x} = np = 60 \times \frac{1}{6} = 10$.

Exercise 103

Find $\bar{x}$ for the binomial distribution $N(q + p)^n$ for:

1. 100 coins tossed, $x = 0, 1, \ldots, 100$ heads.
2. 12 dice cast, $x = 0, 1, \ldots, 12$ aces.
3. 400 patients treated, $x = 0, 1, \ldots, 400$ deaths, if the probability of death is (a) 10%; (b) 20%; (c) 25%.

4. 400 patients treated, $x = 0, 1, \ldots, 400$ survivors, if the probability of death is (a) 10%; (b) 20%; (c) 25%.

5. If $3p = 2q$ and $n = 100$, find p, q, $\bar{x}$.

6. Find n, p, q if $\bar{x} = 20$ and $3p = 2q$.

144. Highest point. In the curve of the binomial distribution $N(q + p)^n$, denote the ordinate at $x = \bar{x}$ by y_0, at $x = \bar{x} + 1$ by y_1, and at $x = \bar{x} - 1$ by y_{-1}. Then, since $\bar{x} = np$ and $p + q = 1$, $np + nq = n$, or $n - np = nq$.

$$y_0 = N\,{}_nC_{np}\, p^{np}\, q^{nq} = N \frac{n!}{(np)!(nq)!} p^{np}\, q^{nq}$$

$$y_1 = N\,{}_nC_{np+1}\, p^{np+1}\, q^{nq-1} = N \frac{n!}{(np+1)!(nq-1)!} p^{np+1} q^{nq-1}$$

$$y_{-1} = N\,{}_nC_{np-1}\, p^{np-1}\, q^{nq+1} = N \frac{n!}{(np-1)!(nq+1)!} p^{np-1}\, q^{nq+1}$$

Let us compare these ordinates by finding the quotients

$$\frac{y_1}{y_0}, \frac{y_{-1}}{y_0}, \frac{y_1}{y_{-1}}.$$

They are

$$(1)\quad \frac{y_1}{y_0} = \frac{(np)!(nq)!}{(np+1)!(nq-1)!}\,\frac{p}{q} = \frac{nq}{np+1} \times \frac{p}{q} = \frac{npq}{npq+q}.$$

$$(2)\quad \frac{y_{-1}}{y_0} = \frac{(np)!(nq)!}{(np-1)!(nq+1)!}\,\frac{q}{p} = \frac{np}{nq+1} \times \frac{q}{p} = \frac{npq}{npq+p}.$$

$$(3)\quad \frac{y_1}{y_{-1}} = \frac{npq}{npq+q} \times \frac{npq+p}{npq} = \frac{npq+p}{npq+q}.$$

Since n, p, q are positive numbers, it is evident from (1) and (2) that $y_0 > y_1$ and also that $y_0 > y_{-1}$. That is, the maximum ordinate is y_0 at $x = \bar{x} = np$ regardless of the relative values of p and q. We also see from (3) that if $p = \frac{1}{2}$, $y_1 = y_{-1}$, that if $p < q$, $y_1 < y_{-1}$, and that if $p > q$, $y_1 > y_{-1}$. Thus the maximum ordinates for the curves on page 234 are at $x = \bar{x} = np = 10 \times \frac{1}{2} = 5$ for A, at $x = 10 \times \frac{1}{4} = 2\frac{1}{2}$ for B, and at $x = 10 \times \frac{3}{4} = 7\frac{1}{2}$ for C.

145. Standard deviation. We have seen that for any frequency distribution

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$

and for a binomial distribution $\bar{x} = np$. The deviations of the separate items from the mean are $x_1 - \bar{x}$, $x_2 - \bar{x}$, and so forth. The squares of these deviations are $(x_1 - \bar{x})^2$, $(x_2 - \bar{x})^2$, and so on, and the sum of these squares for the entire table is $\Sigma f(x - \bar{x})^2$. The mean of the squares of the deviation from $\bar{x}$,

$$\frac{\Sigma f(x-\bar{x})^2}{\Sigma f},$$

is called the *variance.* The square root of the variance, called the *standard deviation,* is represented by the small Greek letter σ (sigma), so that

$$\sigma^2 = \frac{\Sigma f(x-\bar{x})^2}{\Sigma f}$$

and

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}}$$

The standard deviation, σ, is most important in statistical investigations. The calculation of σ from the formula presents no difficulty if the values of $x - \bar{x}$ are integers. But if they are not, the calculation is burdensome. It may, however, be reduced to a relatively simple calculation by following the method used for calculating $\bar{x}$, page 237.

Let $\bar{x} = x' + d$ where x' is an integer near the mean. Then

$$\Sigma f(x-x')^2 = \Sigma f[(x-\bar{x}) + d]^2 = \Sigma f(x-\bar{x})^2 + 2d\Sigma f(x-\bar{x}) + d^2\Sigma f.$$

But since $\Sigma f(x - \bar{x}) = 0$, the relation is

$$\Sigma f(x-x')^2 = \Sigma f(x-\bar{x})^2 + d^2\Sigma f.$$

This relation also shows that the sum of the squares of the deviations from the mean is least. It is less than the sum of the squares of the deviations from any number other than the mean.

To illustrate its use we shall calculate the value of σ for the frequency distribution of the example at the beginning of this chapter.

We take $x' = 67$. The last column, $\Sigma f(x - x' + 1)^2$, is added to the table in order to check the computations. This check is known as *Charlier's check.*

x	f	$x - x'$	$f(x - x')$	$f(x - x')^2$	$f(x - x' + 1)^2$
60	3	−7	− 21	147	108
61	5	−6	− 30	180	125
62	10	−5	− 50	250	160
63	50	−4	−200	800	450
64	107	−3	−321	963	428
65	170	−2	−340	680	170
66	180	−1	−180	180	0
→67	190	0	0	0	190
68	165	1	165	165	660
69	85	2	170	340	765
70	30	3	90	270	480
71	5	4	20	80	125
	1000	−18	−697	4055	3661

Check: $\Sigma f(x-x'+1)^2 = \Sigma f(x-x')^2 + 2\Sigma f(x-x') + \Sigma f.$

$3661 = 4055 - 1394 + 1000 = 3661.$

From $\Sigma f(x-x') = d\Sigma f$, $d = -.697$ and $\bar{x} = 67 - .697 = 66.303.$

From $\Sigma f(x-x')^2 = \Sigma f(x-\bar{x})^2 + d^2\Sigma f.$

$4055 = \Sigma f(x-\bar{x})^2 + 1000\,(-.697)^2.$

$$\sigma^2 = \frac{\Sigma f(x-\bar{x})^2}{\Sigma f} = 4.055 - .4858 = 3.5692.$$

$$\sigma = \sqrt{3.5692} = 1.886.$$

Exercise 104

1. Find the value of σ in examples 3, 4, 5 of exercise 102, page 237.

2. Given the table

x	4	5	6	7	8
f	1	3	8	2	1

(a) Calculate $\bar{x}$ and σ.

(b) Draw the histogram and the frequency polygon.

3. Given the table

x	2	4	5	6	8	9	10
f	1	3	5	8	6	4	2

Find $\bar{x} = 6\frac{17}{29}$ and $\sigma = 1.99$.

4. Calculate $\bar{x}$ and σ for the frequency distribution of example 8, Exercise 99, page 226.

146. The value of σ for a binomial distribution. For any distribution

$$\sigma^2 = \frac{\Sigma f(x-\bar{x})^2}{\Sigma f} = \frac{\Sigma fx^2 - 2\bar{x}\Sigma fx + \bar{x}^2\Sigma f}{\Sigma f}$$

$$= \frac{\Sigma fx^2}{\Sigma f} - \bar{x}^2,$$

since

$$\frac{\Sigma fx}{\Sigma f} = \bar{x}.$$

For the binomial distribution $N(q + p)^n$, fx^2 is written $fx(x - 1) + fx$, so that $x(x - 1)$ may cancel two factors of $x!$ and leave $(x - 2)!$.

$$\Sigma fx^2 = N\Sigma \frac{n!}{x!(n-x)!} p^x q^{n-x} x(x-1) + \Sigma fx$$

$$= Nn(n-1)p^2 \Sigma \frac{(n-2!)}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + Nnp.$$

Since the value of the sigma factor is $(q + p)^{n-2} = 1$,

$$\Sigma fx^2 = Nn(n-1)p^2 + Nnp,$$

$$\frac{\Sigma fx^2}{\Sigma f} = n^2p^2 - np^2 + np,$$

and

$$\sigma^2 = n^2p^2 - np^2 + np - n^2p^2 = np - np^2$$

$$= np(1-p) = npq.$$

That is, the two important constants $\bar{x}$ and σ for a binomial distribution are $\bar{x} = np$, $\sigma = \sqrt{npq}$.

Thus the results for curve A, page 234, are $\bar{x} = 5$, $\sigma = \sqrt{2.5}$; for curve B, $\bar{x} = 2.5$, $\sigma = \sqrt{\frac{30}{16}}$; and for curve C, $\bar{x} = 7\frac{1}{2}$, $\sigma = \sqrt{\frac{30}{16}}$.

Exercise 105

Find $\bar{x}$ and σ for the following.

1. 100 coins are tossed, heads being successes.
2. 360 dice are cast, aces being successes.
3. 360 dice are cast, prime numbers being successes.
4. The distribution represented by $(\frac{5}{6} + \frac{1}{6})^{100}$.
5. The distribution represented by $(\frac{3}{5} + \frac{2}{5})^{1000}$.
6. The theoretical frequencies, f, with which the sums of spots occur, x, when 2 dice are cast 36 times are

x	2	3	4	5	6	7	8	9	10	11	12
f	1	2	3	4	5	6	5	4	3	2	1

and when 3 dice are cast 6^3, or 216, times, the table is

x	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
f	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1

These examples are not binomial distributions. But since each die has 21 spots on it, imagine that it is replaced by 21 dice, each having one spot on one face, the remaining 5 faces being blank. Then for the case of 2 dice we have $(\frac{5}{6} + \frac{1}{6})^{42}$, for which $\bar{x} = 7$ and $\sigma^2 = \frac{35}{6}$.

For 3 dice we have $(\frac{5}{6} + \frac{1}{6})^{63}$ and $\bar{x} = 10\frac{1}{2}$, $\sigma^2 = \frac{35}{4}$. Show that these results are obtained from the frequency tables.

Note: It may be proved that the sum of spots, x, that appear when n ordinary dice are cast, varies from n to $6n$, that $\bar{x} = 21n \times \frac{1}{6}$, and that $\sigma^2 = 21n \times \frac{1}{6} \times \frac{5}{6}$.

147. Significance of σ. We shall see presently that, although the probability curve may extend far to the right and to the left of $x = \bar{x}$, practically the entire area under the curve is included between $x = \bar{x} - 4\sigma$ and $x = \bar{x} + 4\sigma$ and that one half of the total area is included between $x = \bar{x} - \frac{2}{3}\sigma$ and $x = \bar{x} + \frac{2}{3}\sigma$.

Thus if 100 coins are tossed and heads are successes, $n = 100$, $p = q = \frac{1}{2}$, $\bar{x} = 50$, $\sigma = 5$. The ordinates of the probability curve are very short at $x = \bar{x} \pm 4\sigma$, that is, at $x = 70$ and at $x = 30$, and the area between $x = 30$ and $x = 70$ is practically the entire area under the curve. That is, it is almost a certainty that between 30 and 70 heads show when 100 coins are tossed. The area under the curve from $x = 50 - \frac{2}{3} \times 5$ and $x = 50 + \frac{2}{3} \times 5$, or between $x = 47$ and 53, is half the area under the curve. That is, the probability is $\frac{1}{2}$, or the chances are even, that between 47 and 53 heads show when 100 coins are tossed.

148. Normal probability curve. The curve resulting from the binomial distribution $N(q + p)^n$, when n is taken very large, is called the *normal probability curve*, and its equation reduces to the relatively simple form

$$y = \frac{N}{\sigma} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}t^2},$$

where

$$t = \frac{x - \bar{x}}{\sigma}, \; e = 2.718\ldots, \; \pi = 3.14159\ldots.$$

This equation is perhaps the most important relation in the entire mathematical theory of statistics. It may be derived as follows.

Let the ordinate at $x = \bar{x} = np$ be designated by y_0 and the ordinate at $x = \bar{x} + k = np + k$ by y_k. Then

$$y_0 = N {}_nC_{np}\, p^{np} q^{nq} = N \frac{n!}{(np)!(nq)!} p^{np} q^{nq}$$

$$y_k = N {}_nC_{np+k}\, p^{np+k} q^{nq-k} = N \frac{n!}{(np+k)!(nq-k)!} p^{np+k} q^{nq-k}$$

$$\frac{y_k}{y_0} = \frac{(np)!(nq)!}{(np+k)!(nq-k)!} \frac{p^k}{q^k}$$

$$= \frac{(nq)\,(nq-1)\,(nq-2) \quad \ldots\,(nq-k+2)\,(nq-k+1)}{(np+k)\,(np+k-1)\,(np+k-2)\ldots \quad (np+2)\,(np+1)} \times \frac{p^k}{q^k}.$$

There are k factors in the numerator and also in the denominator. Take out of each factor of the numerator the factor nq, and out of each factor of the denominator the factor np. Then, since

$$\frac{(nq)^k}{(np)^k}\frac{p^k}{q^k} = 1,$$

the relation becomes

$$\frac{y_k}{y_0} = \frac{1\cdot\left(1-\frac{1}{nq}\right)\left(1-\frac{2}{nq}\right)\ldots\left(1-\frac{k-2}{nq}\right)\left(1-\frac{k-1}{nq}\right)}{\left(1+\frac{k}{np}\right)\left(1+\frac{k-1}{np}\right)\left(1+\frac{k-2}{np}\right)\ldots\left(1+\frac{2}{np}\right)\left(1+\frac{1}{np}\right)}.$$

$\text{Log}_e\left(\frac{y_k}{y_0}\right)$ consists of k terms such as $\log\left(1-\frac{3}{nq}\right)$ and k terms such as $\log\left(1+\frac{5}{np}\right)$, none of the fractions exceeding $\frac{k}{nq}$ or $\frac{k}{np}$.

We may take n as large as we please. For fixed values of p and q, and for a wide range of values of k, n may be taken sufficiently large so that $\frac{k}{nq}$ and $\frac{k}{np}$ will each be less than a small fraction, say .001.

The relation

$$\log_e(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \ldots,$$

page 184, shows that, for values of z as small as .001, the second and higher powers of z may be disregarded, so that, approximately,

$$\log_e(1+z) = z,$$

and

$$\log_e(1-z) = -z.$$

Then

$$\log_e\left(\frac{y_k}{y_0}\right) = \left[0-\frac{1}{nq}-\frac{2}{nq}-\ldots-\frac{k-1}{nq}\right] - \left[\frac{1}{np}+\frac{2}{np}+\ldots+\frac{k}{np}\right].$$

The expressions in brackets are AP's and

$$\log_e\left(\frac{y_k}{y_0}\right) = -\frac{k(k-1)}{2nq} - \frac{k(k+1)}{2np} = -\frac{k^2}{2npq} + \frac{k(p-q)}{2npq}.$$

In the binomial distribution $(q+p)^n$, $\sigma^2 = npq$, and, since k is measured from the mean, $\bar{x}$, its value is $x - \bar{x}$. If k is measured in σ units so that

$$\frac{k}{\sigma} = \frac{x-\bar{x}}{\sigma} = t,$$

$$\log_e\left(\frac{y_k}{y_0}\right) = -\tfrac{1}{2}t^2 + \tfrac{1}{2}\frac{t(p-q)}{\sigma}.$$

If $p = q = \frac{1}{2}$,

$$\log_e\left(\frac{y_k}{y_0}\right) = -\tfrac{1}{2}t^2,$$

and

$$y_k = y_0e^{-\frac{1}{2}t^2}.$$

If p and q are unequal, $p - q$ is numerically less than 1. Since, for large values of n, σ is also large, $\dfrac{p-q}{2\sigma}$ is a small fraction. The value of $-\frac{1}{2}t^2$ for $t = 4$ is -8, and the addition of the small fraction $\dfrac{t(p-q)}{2\sigma}$ to $-\frac{1}{2}t^2$ will not change the value of $-\frac{1}{2}t^2$ appreciably. We conclude therefore that, when n is very large, whether p and q are equal or unequal, and for values of t as large as 4 or 5, the relation is approximately

$$\log_e\left(\frac{y_k}{y_0}\right) = -\tfrac{1}{2}t^2, \text{ or } y_k = y_0e^{-\frac{1}{2}t^2}.$$

The graph of this equation is shown in Fig. 34 by the curves ABC and DEF from the following table:

t	$\frac{1}{2}t^2$	$e^{-\frac{1}{2}t^2}$	t	$\frac{1}{2}t^2$	$e^{-\frac{1}{2}t^2}$
.0	.00	1.0000	2.2	2.42	.0889
.2	.02	.9802	2.4	2.88	.0561
.4	.08	.9231	2.6	3.38	.0340
.6	.18	.8353	2.8	3.92	.0198
.8	.32	.7261	3.0	4.50	.0111
1.0	.50	.6065	3.2	5.12	.0060
1.2	.72	.4868	3.4	5.76	.0032
1.4	.98	.3753	3.6	6.48	.0015
1.6	1.28	.2780	3.8	7.22	.0007
1.8	1.62	.1979	4.0	8.00	.0003
2.0	2.00	.1353	4.2	8.82	.0001

The curve ABC is drawn so that $y_0 = OB = 1$, and the curve DEF so that $y_0 = OE = .3989$.

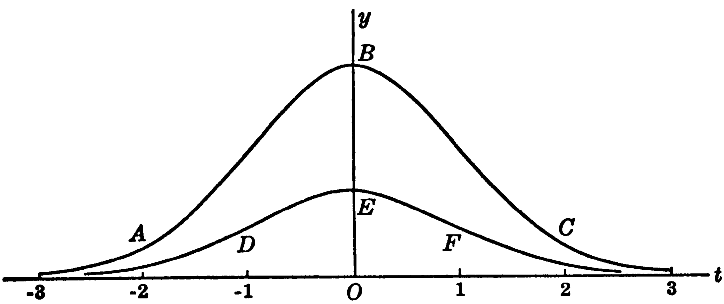


Fig. 34.

The area under the curve, found by integration, is $y_0\sigma\sqrt{2\pi}$. But since this area is represented by the same number, N, as the total frequency,

$$y_0\sigma\sqrt{2\pi} = N, y_0 = \frac{N}{\sigma\sqrt{2\pi}},$$

and the final equation is

$$y = \frac{N}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\bar{x})^2/\sigma^2}.$$

$\frac{1}{\sqrt{2\pi}} = .3989$, and the curve DEF, Fig. 34, is drawn so that the ordinates are .3989 of the ordinates of the curve ABC, thereby making the area under the curve DEF equal to a unit, 1.

149. Area under a curve. In the preceding article, the values of $e^{-\frac{1}{2}t^2}$ are easily calculated to any degree of accuracy from the expansion

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

by setting $z = -\frac{1}{2}t^2$.

The method of finding the area under a curve by integration is essentially a summation of rectangles symmetrically drawn about ordinates that are equally spaced. The smaller the interval between two consecutive ordinates, the more nearly is the sum of the rectangles the same as the area under the curve.

Illustration

Find the area under the curve $y = x^2 - 2x + 2$ from $x = 3$ to $x = 4$.

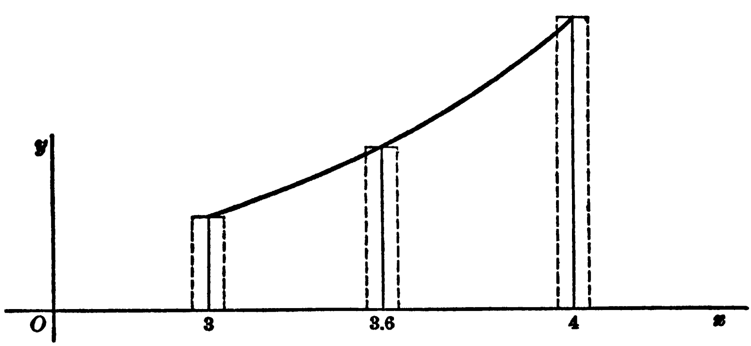


Fig. 35.

Divide the distance from $x = 3$ to $x = 4$ into any number of equal parts, say 10. At each division point, such as at $x = 3.6$, draw the

ordinate. Draw a rectangle symmetrical about the ordinate and make its width .1, and do the same for each ordinate. The lengths of the ordinates calculated from the given equation are

x	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0
y	5.00	5.41	5.84	6.29	6.76	7.25	7.76	8.29	8.84	9.41	10.00,

and their sum is 80.85. The sum of the areas of the rectangles is $80.85 \times .1 = 8.085$. But half of the rectangle at $x = 3$ and half of the rectangle at $x = 4$ are not under the curve. The sum of these two rectangles is $\frac{1}{2}(5+10)\ (.1) = .75$. Hence the area under the curve is approximately $8.085 - .75 = 7.335$. The exact area, found by integration, is $7\frac{1}{3}$.

In general, to find the area from $x = a$ to $x = b$, under a curve whose equation is given, divide the distance $b - a$ into n equal parts, each of which is $h = \dfrac{b-a}{n}$. The ordinates at the division points are indicated by y_0, y_1, y_2, . . . y_n, where y_0 is the ordinate at $x = a$ and y_n is the ordinate at $x = b$. The area under the curve is then given approximately by

$$\text{Area} = h[\Sigma y - \tfrac{1}{2}(y_0 + y_n)].$$

The smaller h is made, the more nearly will the area calculated by this formula agree with the area found by integration.

Exercise 106

Find the areas under the following curves between the limits indicated by using the appropriate tables.

1. $y = x^{1/2}$ from $x = 1$ to $x = 9$, taking intervals of 1.
2. $y = x^{1/3}$ from $x = 1$ to $x = 8$, taking intervals of 1.
3. $y = \log_{10} x$ from $x = 3$ to $x = 4$, taking intervals of .1.
4. $y = e^x$ from $x = 0$ to $x = 2$, taking intervals of .1.
5. $y = e^{-x}$ from $x = 0$ to $x = 2$, taking intervals of .1.
6. $y = e^{-\frac{1}{2}t^2}$ from the table on page 247, from $t = -3$ to $t = +3$.

7. Show that $\sqrt{2\pi} = 2.5066$ and $\dfrac{1}{\sqrt{2\pi}} = .3989$.

150. Table of ordinates and areas. Table XVI gives the values of the ordinates and of the areas under the curve

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = .3989\, e^{-\frac{1}{2}t^2}$$

for values of t from 0 to 4 at intervals of .05 calculated by the method shown in the preceding article.

The equation is obtained from that of the normal probability curve (p. 247)

$$y = \frac{N}{\sigma} \cdot \frac{1}{\sqrt{2\pi}}\; e^{-\frac{1}{2}(x-\bar{x})^2/\sigma^2}$$

as follows: Since the ordinates are proportional to N, the value of N is taken as 1, and therefore the total area is 1, the area of the part to the right of $x = \bar{x}$ being .5000. Then $\dfrac{x - \bar{x}}{\sigma}$ is replaced by t so that t measures the deviations of x from the mean $\bar{x}$ in units σ instead of the ordinary units 1 pound, 1 inch, and so forth.

According to the table, the area from $t = 0$ to $t = 3$, that is, from $x = \bar{x}$ to $x = \bar{x} + 3\sigma$, is .4987, or 99.74% of the area under half of the curve. That is, the probability that x lies between $x \pm 3\sigma$ is practically a certainty.

The value of t for which the area is .2500 is, by interpolation, $t = .6745$, or t is approximately $\frac{2}{3}$. That is, it is an even chance that any x taken at random will be between $\bar{x} \pm \frac{2}{3}\sigma$.

Thus if 100 coins are tossed and heads are successes indicated by x, $n = 100$, $p = q = \frac{1}{2}$, $\bar{x} = 50$, $\sigma = 5$. $\bar{x} \pm 3\sigma = 50 \pm 15$; $\bar{x} \pm \frac{2}{3}\sigma = 50 \pm 3$.

It is almost a certainty that between 35 and 65 heads show; and it is an even chance that between 47 and 53 heads show.

151. Use of the normal curve. Probability calculations in cases of binomial distributions of frequencies are made by using the ideal normal curve as in the following

Illustrations

In a large number of hospital cases treated in a certain manner, 25% of the patients die within a year.

1. Find the probability, P, that, of 48 cases treated, exactly 12 deaths occur.

$$\text{Here } n = 48, p = \tfrac{1}{4}, q = \tfrac{3}{4}, \text{and } P = {}_{48}C_{12}(\tfrac{1}{4})^{12}(\tfrac{3}{4})^{36} = \frac{48!3^{36}}{36!12!4^{48}} = .132$$

by using logarithms. The probabilities of 0, 1, 2, . . . deaths give a frequency distribution which is not continuous but is discrete, and the probability of 12 deaths is the area under the curve from $11\frac{1}{2}$ to $12\frac{1}{2}$. In this case, $n = 48$, $p = \frac{1}{4}$, $q = \frac{3}{4}$ give $\bar{x} = np = 12$ and $\sigma = \sqrt{npq} = 3$. The deviations of $11\frac{1}{2}$ and $12\frac{1}{2}$ from the mean, 12, are $-\frac{1}{2}$ and $+\frac{1}{2}$, which in σ units are $t = -\frac{1}{6}$ and $t = +\frac{1}{6}$. The probability of 12 deaths is the area under the normal curve from $t = -\frac{1}{6}$ to $t = +\frac{1}{6}$, or twice the area from $t = 0$ to $t = \frac{1}{6}$, which is .1322.

2. Find the probability that exactly 8 deaths occur. Here

$$P = {}_{48}C_8\,(\tfrac{1}{4})^8(\tfrac{3}{4})^{40} = \frac{48!3^{40}}{40!8!4^{48}} = .0579$$

by logarithms. With the probability curve, the area under the curve is required from $7\frac{1}{2}$ to $8\frac{1}{2}$, which deviate from the mean, 12, by -4.5 and -3.5, or in σ units by -1.5 and $-1.16\frac{2}{3}$. Since the curve is symmetrical about $t = 0$, the required probability is the area under the curve from $t = 1.16\frac{2}{3}$ to $t = 1.50$, which is .0550.

3. Find the probability that less than 15 deaths occur. By the binomial distribution method, it would be necessary to calculate the separate probabilities of 0, 1, 2, . . . , 14 deaths and to add the results, obviously a very lengthy calculation. By using the normal curve, it is only necessary to calculate the area under the curve from 0 to $14\frac{1}{2}$ deaths. The deviations from the mean, 12, are -12 and $+2\frac{1}{2}$, or, in σ units, -4 and $+\frac{5}{6}$. The area under the curve from $t = -4$ to $t = +\frac{5}{6}$ is $.5000 + .2976 = .7976$.

4. A new treatment of the disease resulted in 10 deaths in 48 cases treated, whereas 12 deaths might have been expected from the old treatment. Can any definite conclusion be drawn from this result?

The probability of less than 10 deaths is the area under the curve from 0 to $9\frac{1}{2}$, or from $t = -12/3$ to $t = -2.5/3$, an area of $.5000 - .2976 = .2024$ and a probability of 20.24%. In 100 sets of 48 cases each, under the old treatment, less than 10 deaths occur approximately 20 times, or about once in five sets, so that this result can hardly be ascribed to the new method of treatment. Furthermore, the probability of a deviation of 2 from the mean, 12, in either direction is the area under the normal curve from $t = -\frac{2}{3}$ to $t = +\frac{2}{3}$, or approximately .5000. That is, as a matter of pure chance, it is a 1 to 1 chance that in any set of 48 cases treated, the number of deaths will be between 10 and 14.

Exercise 107

1. Find the area under the normal curve (a) from $t = 0$ to $t = 1$; (b) from $t = 1$ to $t = 2$; (c) from $t = 2$ to $t = 3$; (d) from $t = 1.27$ to $t = 1.43$.

2. Find the value of t if the area under the normal curve to the right or left is .3333 (a) from $t = 0$; (b) from $t = .50$; (c) from $t = 1.50$.

3. Find the probability that when 100 coins are tossed (a) exactly 40 heads show; (b) not more than 40 heads show; (c) at least 40 heads show.

4. Find the probability that when 180 dice are cast (a) exactly 35 aces show; (b) not more than 35 aces show.

5. If the batting average of a baseball player is .300, find the probability that in 50 times at bat he will get (a) exactly 10 hits; (b) at least 10 hits; (c) at most 10 hits.

6. According to the mortality table, $l_{41} = 77341$, $d_{41} = 774$, the probability of death within a year being 1%. Find the probability that, of 10,000 persons 41 years of age, (a) exactly 90 will die within 1 year; (b) not more than 90 will die within 1 year; (c) at least 90 will die within 1 year.

152. Fitting a normal curve to given data. The frequency distribution in the example at the beginning of this

chapter is sufficiently like a normal distribution to warrant the use of the equation

$$y = \frac{N}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\bar{x})^2/\sigma^2}.$$

In our problem, $N = 1000$, $\bar{x} = 66.303$, $\sigma^2 = 3.569$,

$$\sigma = 1.889, \frac{N}{\sigma} = \frac{1000}{1.889} = 529.$$

The relation between f and x of the given table is therefore

$$f = 529 \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-66.303)^2/3.569} = 211\, e^{-(x-66.303)^2/7.138}.$$

To see how values of f calculated from this equation agree with the given values, the tabulation is:

x	$x - \bar{x}$	$\frac{x-\bar{x}}{\sigma} = t$	$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$	$f = y \times 529$	f in the table
60	−6.303	−3.34	.0015	1	3
61	−5.303	−2.81	.0077	4	5
62	−4.303	−2.28	.0296	16	10
63	−3.303	−1.75	.0863	46	50
64	−2.303	−1.22	.1895	100	107
65	−1.303	− .69	.3144	166	170
66	− .303	− .16	.3939	208	180
67	.697	.37	.3727	198	190
68	1.697	.90	.2661	141	165
69	2.697	1.42	.1456	77	85
70	3.697	1.95	.0596	32	30
71	4.697	2.48	.0184	10	5

The student may fit a normal curve to the frequency distribution of example 8, Exercise 99, page 226.

CHAPTER XIII

TRANSFORMATION OF EQUATIONS

153. Equation and graph. In the study of statistical data, we are confronted with sets of corresponding numbers which may be the results of human endeavor or of the operations of the laws of nature. Because of the many factors involved, it is not to be expected that the numerical data obey a law that can be expressed by a mathematical equation. Nevertheless, an equation is desirable even though it gives a relation that only approximates the relation that actually exists.

Before attempting to find approximate relations, it is necessary to know how an equation and its graph are related, how the graph is drawn when the equation is given, and how the equation may be found when the graph is given.

From an equation that shows the relation between the variables x and y, a table of corresponding values of x and y can be calculated. Each pair of corresponding values is represented graphically by a point on a plane, and the aggregate of all the points forms a geometric figure, the graph of the equation. The graph shows not only the points given by the table but all intermediate points and points beyond the table as well. That is, the graph is a continuous curve.

Different forms of equations correspond to different kinds of curves. Peculiarities in the curve correspond to algebraic relations inherent in the equation.

154. Symmetry. A curve is drawn on a sheet of paper. If the paper is folded along a line and the two parts of the

curve into which the fold divides it coincide, the curve is said to be symmetrical about the line of fold.

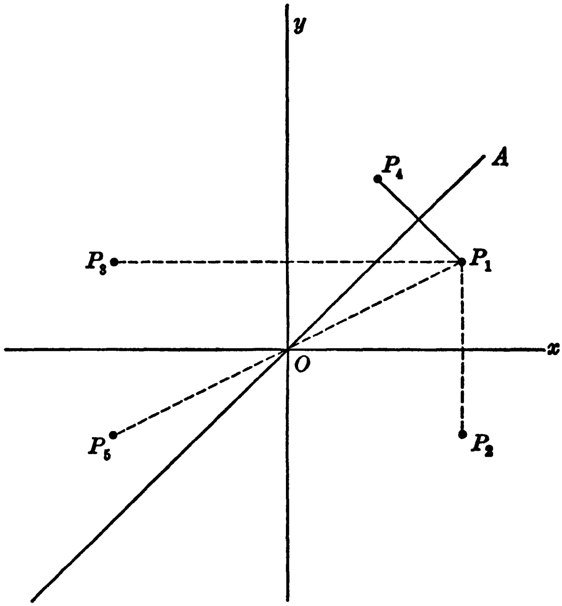


Fig. 36.

Let P_1 (x, y) be any point of a curve whose equation is given (Fig. 36).

For symmetry about the x axis, P_2 $(x, -y)$ must be another point of the curve; for symmetry about the y axis, P_3 $(-x, y)$ must be another point of the curve; for symmetry about the 45° line OA, the line through the origin bisecting the angle xOy, P_4 (y, x) must be another point of the curve.

Symmetry about the origin, O, means that to the point P_1 there corresponds another point P_5, such that on the straight line P_1OP_5, $OP_5 = OP_1$, and the coördinates of P_5 are $(-x, -y)$.

Therefore the equation of a curve can be tested for symmetry as follows:

For symmetry about	*the* x *axis*	y *axis*	*origin*	*the 45° line*
Replace	y by $-y$ and x by x	x by $-x$ and y by y	x by $-x$ and y by $-y$	x by y and y by x

Then if the substitution leaves the equation unchanged, the symmetry tested exists.

Illustrations

1. $x^2 + y^2 = 9$. There are all 4 kinds of symmetry.

2. $x^2 + xy + y^2 = 4$. There is symmetry about the origin and also about the 45° line.

3. $x^2 + xy - y^2 = 4$. There is symmetry about the origin only.

4. $xy + x + y = 0$. There is symmetry about the 45° line only.

5. $y = x^3 - 2$. Symmetry is not evident.

Exercise 108

State the kind of symmetry to be found in the graphs of each of the following equations.

1. $y = x$
2. $y = x^2$
3. $y^2 = x$
4. $y = x^3$
5. $y^2 = x^3$
6. $y^3 = x^2$
7. $xy = 10$
8. $xy = x + y$
9. $y^2 = x - 4$
10. $xy^2 = 10$

155. Intercepts and limits. The distances from the origin that the curve cuts off on the coördinate axes are called the *intercepts*. The x intercept is found by setting $y = 0$, and the y intercept by setting $x = 0$. The curve passes through the origin if the equation is satisfied by $x = 0$, $y = 0$.

The solution of the equation for x or for y usually enables

us to set limits to the curve. Thus, if we solve the equation $x^2 + xy + y^2 = 4$ for y, we obtain

$$y = \frac{-x \pm \sqrt{16-3x^2}}{2}.$$

Now if $3x^2 > 16$, $16 - 3x^2$ is negative, $\sqrt{16 - 3x^2}$ is imaginary, and the graph does not exist. Hence in this case the graph can exist only in the region from $x = -\sqrt{\frac{16}{3}}$ to $x = +\sqrt{\frac{16}{3}}$.

If the equation is $xy + x + y = 0$,

$$y = \frac{-x}{x+1}.$$

For the value $x = -1$, $y = \frac{1}{0}$. There is no point on the curve that corresponds to $x = -1$ since $\frac{1}{0}$ has no meaning, but values of y can be found for values of x that are near -1, and these values become numerically larger as x is taken closer to -1. The line $x = -1$ is an *asymptote* of the curve. The curve approaches this line as closely as we please but never reaches it.

Exercise 109

Find the intercepts and the limits for graphs of the following equations and sketch the graphs.

1. $x^2 + y^2 = 9$
2. $y^2 = 2x - 10$
3. $x^2 = 2y - 10$
4. $x^2 - 2x + y - 3 = 0$
5. $xy - 2x - 3y = 5$
6. $x^2 - xy - 2y^2 + 4 = 0$

156. Translation of axes. A curve may be symmetrical about a horizontal line other than the x axis, about a vertical line other than the y axis, or about a point other than the origin, and such symmetry can be discovered. The process of moving the origin to a different point and taking new axes parallel to the original axes is called *translating the axes.* When the axes are translated, the

curve is not changed but the equation of the curve is changed. Let QPR be any curve and P any point on it. If the curve is referred to the axes Ox and Oy, the coordinates of P are (x, y). Now translate the axes to $O'x'$ and $O'y'$ parallel to Ox and Oy, so that the new origin,

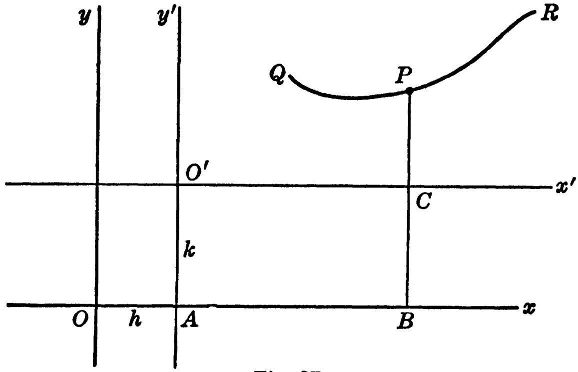


Fig. 37.

O', has the coördinates (h, k) with reference to the original axes. With reference to the new axes, the coördinates of P are (x', y'). It is now apparant from the diagram that

$$x = OB = OA + AB = OA + O'C = h + x',$$

and

$$y = BP = BC + CP = AO' + CP = k + y'.$$

That is, to translate the axes so that the new origin is at (h, k), replace x by $h + x'$ and y by $k + y'$.

Illustrations

Transform the equation $y = 3x^2 - 12x + 8$ by translating the axes so that (a) the new equation shall not have a constant term; (b) symmetry shall be evident.

Let the new origin be at (h,k). Then the new equation is

$$k + y' = 3\,(h + x')^2 - 12\,(h + x') + 8,$$

or

$$y' = 3x'^2 + x'\,(6h - 12) + (3h^2 - 12h + 8 - \mathrm{k}).$$

(a) Set $3h^2 - 12h + 8 - k$ equal to zero, and the transformed equation is

$$y' = 3x'^2 + x'(6h - 12).$$

Any values may be given to h and k provided $3h^2 - 12h + 8 - k = 0$. Thus if $h = 1$ and $k = -1$, the origin is moved to $(1, -1)$, and the transformed equation is

$$y' = 3x'^2 - 6x'.$$

(b) Symmetry will be evident if the constant term and the x' term can be made to vanish.

Set $3h^2 - 12h + 8 - k = 0$ and also $6h - 12 = 0$ and obtain the equation $y' = 3x'^2$. This equation shows symmetry about the new y axis, $O'y'$. The coördinates of the new origin, (h, k), found by solving the simultaneous equations

$$3h^2 - 12h + 8 - k = 0$$

and

$$6h - 12 = 0,$$

are $h = 2$, $k = -4$. Therefore the axes should be translated by moving the origin to $(2, -4)$.

Exercise 110

1. Transform the following equations by translating the axes:

(a) $(x+2)^2 + (y-5)^2 = 25$, new origin $(-2, 5)$
(b) $y = 6x - 12$, new origin $(2, 0)$
(c) $(x+4)(y-3) = 18$, new origin $(-4, 3)$
(d) $(y-4)^2 + 3(x-2) = 9$, new origin $(2, 4)$
(e) $x^3 - x = y + 2$, new origin $(0, -2)$

2. State the kind of symmetry in each of the graphs of example 1.

3. Transform the following equations by translating the axis so as to make symmetry evident, and state the kind of symmetry.

(a) $xy + 3y = 18$
(b) $xy - 2x = 12$
(c) $xy + 2x - 3y = 24$
(d) $y + x^2 = 2$
(e) $y = x^2 + x$
(f) $y = 2x^2 - 5x + 1$
(g) $x = y^2 - 2$
(h) $x = y^2 - 3y$
(i) $x = y^2 - 3y + 2$
(j) $x^3 = y - 2$
(k) $x^3 - 2x + 5 = y$
(l) $y = 3^{x^2-4}$

157. The logarithmic scale. The result of translating the axes is an equation that has the same algebraic form as the original equation but differs from it in that the coefficients are different. An equation can also be transformed so as to produce a totally different algebraic form by marking on the coördinate axes scales other than the algebraic scale of numbers. Many different scales may be used.

In Fig. 38, S is the standard scale, the scale commonly used in making graphic representations. It consists of equal space divisions, and a number on this scale, say 3 at P, indicates that P is 3 unit spaces from the origin O. (A unit space may represent 1000 number units or .01 of a number unit if we so desire.)

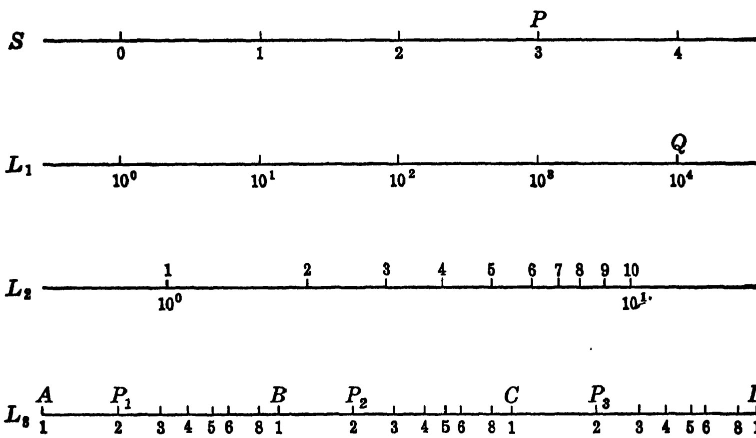


Fig. 38.

On the logarithmic scale L_1, the *logarithm* of the number at a division point indicates the number of unit spaces from O. Thus the number at Q is $10^4 = 10{,}000$, and Q is 4 spaces or log 10^4 from O. Scale L_2 shows how a unit

space on the logarithmic scale is subdivided. The subdivision requires the use of the table of logarithms:

$\log 2 = .3010$	$\log 4 = .6021$	$\log 6 = .7781$
$\log 3 = .4771$	$\log 5 = .6990$	$\log 7 = .8451$
$\log 8 = .9031$	$\log 9 = .9542$	

Scale L_3 shows how the unit spaces AB, BC, CD are subdivided.

Since $\log 2 = .3010$, the figure 2 is placed at the points P_1, P_2, P_3, which are .3010 of the unit space measured from A, B, C. Since $\log 4 = 2 \log 2$, the figure 4 is twice as far from A, B, or C as the figure 2; $\log 8 = 3 \log 2$, and 8 is three times as far from A, B, or C as 2.

If A is marked 1, B is 10, C is 100, D is 1000, P_1 is 2, P_2 is 20, and so on. If B is marked 1, C is 10, D is 100, A is .1, P_1 is .2, and so on. If C is marked 1, D is 10, B is .1, A is .01, P_1 is .02, and so on.

Since there are no logarithms for zero or for negative numbers, 0 and minus numbers do not appear on the logarithmic scale L, and the origin, whether A, B, or C, is marked 1.

The number x is written at a division point X spaces from the origin, and the relation between x and X is $\log x = X$, or $x = 10^X$.

158. Ruled paper. Three kinds of ruled paper, obtainable at any good stationery store at very low cost, should be used freely.

(1) Standard, ordinary, or square ruled paper is equally spaced horizontally and vertically (S S paper).

(2) Semilogarithmic or semilog paper is ruled so that the standard scale may be marked on the horizontal axis and the logarithmic scale on the vertical axis (S L paper, Fig. 39, page 266). It may, of course, also be used so that

the logarithmic scale is on the horizontal axis and the standard scale on the vertical axis (L S paper).

(3) Logarithmic paper is ruled so that the logarithmic scale may be marked on both axes (L L paper, Fig. 40, page 267).

A given table of corresponding values of x and y may be plotted as points on ruled paper on which the scales on the coördinate axes are S, S; S, L; L, S; L, L.

The geometric figure outlined by the aggregate of points will generally be different when we change the scales on the coördinate axes.

A number of other scales may also be used. Thus the successive division points on standard ruled paper may be marked $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and so forth, or 0^2, 1^2, 2^2, 3^2, 4^2, and so on, or 10^{10^0}, 10^{10^1}, 10^{10^2}, and so forth. Such scales are useful, but we shall not discuss them.

159. The straight line on ordinary ruled paper. The most important characteristic of a straight line is that it has a constant slope, s, defined on page 81 by

$$s = \frac{\text{rise}}{\text{advance}}.$$

The equation of the line is $y = sx + b$, where b is the y intercept (see page 85).

This linear relation is fundamental and may be written at sight.

Illustration

x	2	5	11	20	26
y	3	5	9	15	19

As we proceed from one point to another, the slope is always $\frac{2}{3}$, or $s = \frac{2}{3}$. If (x, y), any point on the line, is joined with $(2, 3)$, the rise is $y - 3$ and the advance is $x - 2$. Hence the slope is also

$$\frac{y - 3}{x - 2}$$

and the relation is

$$\frac{y-3}{x-2} = \frac{2}{3}, \text{ or } 3y - 2x = 5.$$

From this relation, the y value of any point of the line may be calculated.

Exercise 111

Write the equations of the straight lines for those of the following tables in which there are linear relations between x and y.

1.

x	3	5	9	15	19
y	2	5	11	20	·26

2.

x	5	7	8	10	14
y	7	9	10	12	16

3.

x	16	13	10	7	4
y	2	4	6	8	10

4.

x	5	10	20	40	80
y	3	7	15	31	63

5.

x	a	$a+c$	$a+2c$	$a+3c$	$a+4c$
y	b	$b+d$	$b+2d$	$b+3d$	$b+4d$

6.

x	log 5	log 10	log 20	log 40	log 80
y	3	7	11	15	19

7.

x	3	7	11	15	19
y	log 2	log 4	log 6	log 8	log 10

8.

x	log 5	log 15	log 45	log 135	log 405
y	log 2	log 4	log 8	log 16	log 32

9. In example 3, find the x value when $y = 15$, and also the y value when $x = 20$.

10. In example 6, find the x value when $y = 4$, and also the y value when $x = \log 35$.

160. The straight line on any kind of ruled paper. Let the coördinates of any point P of the line be designated by (X, Y), which represent the numbers of *unit spaces* from the origin measured along the coördinate axes, and let the same point be designated by (x, y), which represent the

actual *numerical* values of the coördinates of P. Then the equation of a straight line in terms of unit spaces is

$$Y = sX + b,$$

But in terms of x, y, the relation depends upon the scales used or upon the relations between x and X and between y and Y. (1) On (S S), or ordinary ruled paper, $X = x$, $Y = y$, and the relation is

$$y = sx + b,$$

(2) On semilog paper (S L), $X = x$, $Y = \log y$, and the relation is

$$\log y = sx + b.$$

Written without using logs, this relation takes the form

$$y = 10^{sx+b} = 10^{sx}\,10^{b} = (10^{s})^{x}\,(10^{b}).$$

Since s and b are constants, 10^{s} and 10^{b} are also constants and may be replaced by h and k. The form accordingly is

$$y = h^{x}k.$$

(3) On semilog paper (L S), the form of relation is

$$x = h^{y}k.$$

(4) On log paper (L L), $X = \log x$, $Y = \log y$, and the relation is

$$\log y = s \log x + b.$$

This form is changed to

$$\log y - \log x^{s} = b;$$

then to

$$\log\left(\frac{y}{x^{s}}\right) = b;$$

then to

$$\frac{y}{x^{s}} = 10^{b};$$

and finally to

$$y = x^{h}k,$$

where $h = s$ and $k = 10^{b}$.

Illustrations

Two points $A(3, 10)$ and $B(5, 7)$ are plotted on the four kinds of ruled paper and joined by a straight line. Find the relation between x and y for any point of the line in each case.

(1) On (S S) paper, $s = -\frac{3}{2}$ and also $(y - 10)/(x - 3)$. Hence the relation is $2y + 3x = 29$.

(2) On (S L) paper, the form of relation is $y = h^x k$, and h and k must be determined.

Since both of the given points must fit this relation,

$$10 = h^3 k$$

and

$$7 = h^5 k.$$

Eliminate k by division, $\frac{7}{10} = h^2$, or $h = \left(\frac{7}{10}\right)^{1/2}$.

Then

$$k = \frac{10}{h^3} = 10\left(\frac{7}{10}\right)^{-3/2}.$$

The relation is

$$y = 10\left(\frac{7}{10}\right)^{(x-3)/2}.$$

(3) On (L S) paper, the form is $x = h^y k$, and the final relation is

$$x = 3\left(\frac{3}{5}\right)^{(y-10)/3}.$$

(4) On (L L) paper, the form is $y = x^h k$. The equations from which h and k are found are

$$10 = 3^h k$$

and

$$7 = 5^h k.$$

Hence

$$\frac{10}{7} = \left(\frac{3}{5}\right)^h$$

and

$$k = \frac{10}{3^h}.$$

The relation between x and y is:

$$y = x^h \frac{10}{3^h} = 10\left(\frac{x}{3}\right)^h.$$

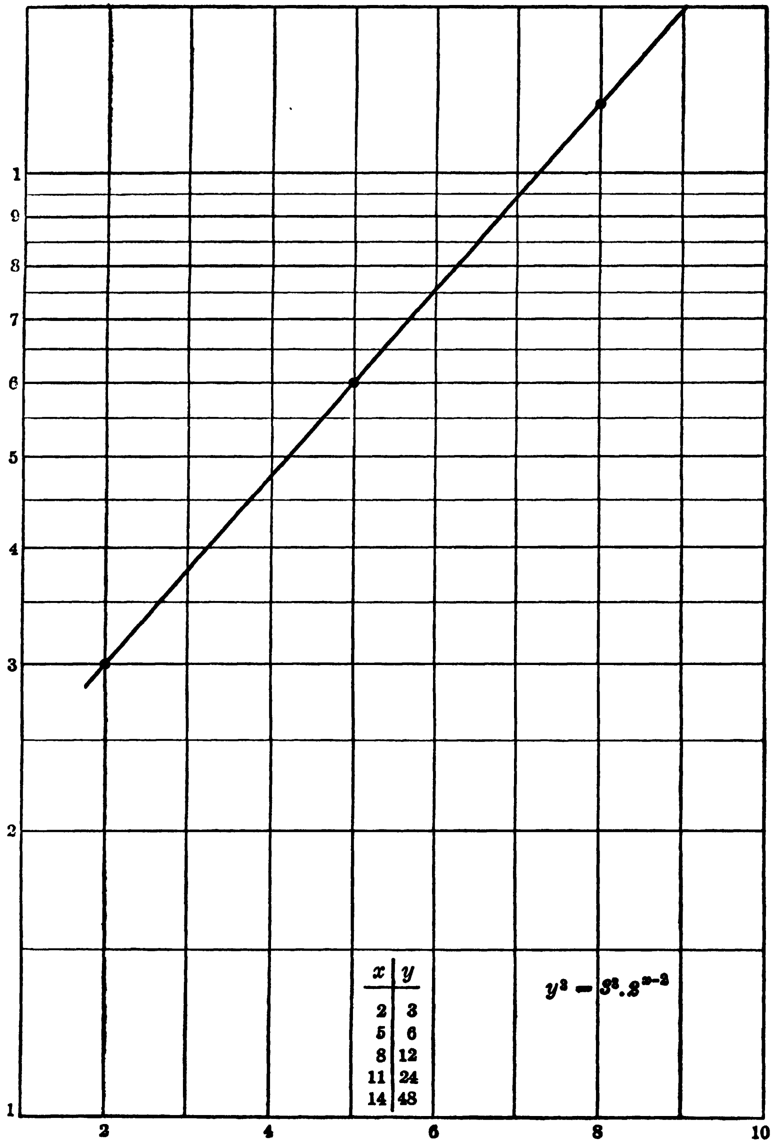
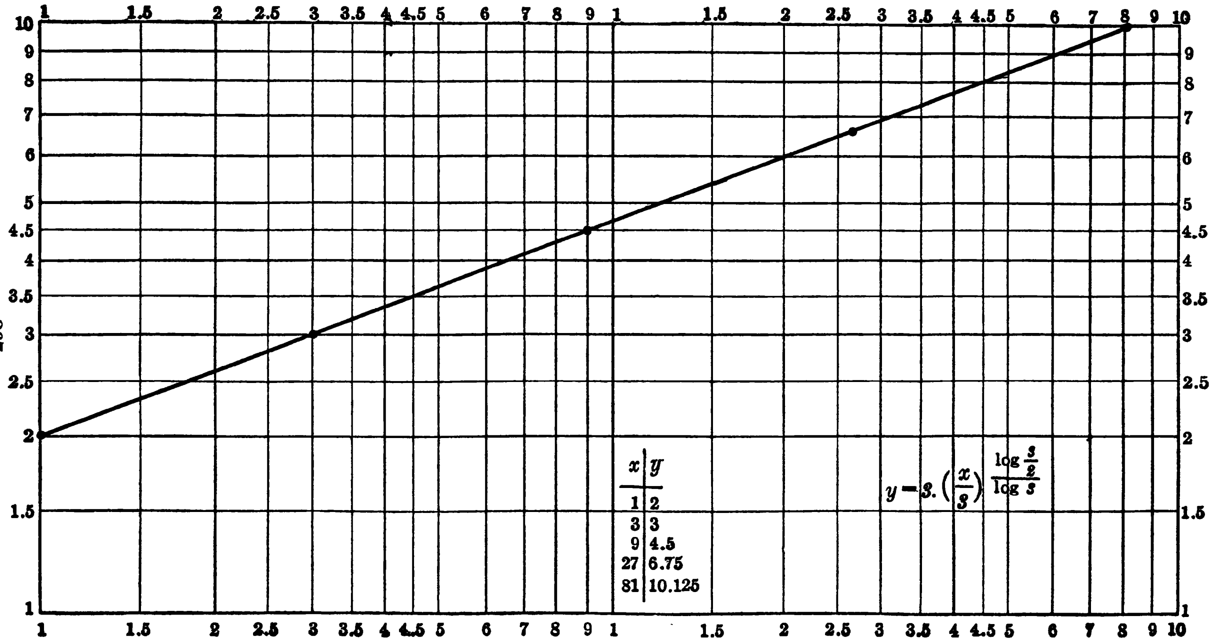


Fig. 39.



$y = 3.\left(\frac{x}{3}\right)^{\frac{\log \frac{3}{2}}{\log 3}}$

x	y
1	2
3	3
9	4.5
27	6.75
81	10.125

Fig. 40.

But from $\frac{10}{7} = (\frac{3}{5})^h$,

$$h = \log \tfrac{10}{7} / \log \tfrac{3}{5}.$$

Hence

$$y = 10\left(\frac{x}{3}\right)^{\log \frac{10}{7} / \log \frac{3}{5}.}$$

Upon attempting to check the values $x = 5$, $y = 7$, we find

$$7 = 10\left(\frac{5}{3}\right)^{\log \frac{10}{7} / \log \frac{3}{5}.}$$

This is a true relation since it may be written

$$\frac{7}{10} = \left(\frac{5}{3}\right)^{\log \frac{7}{10} / \log \frac{5}{3}},$$

and we saw (page 111) that

$$a^{\log c / \log a} = c.$$

Exercise 112

A straight line joins the points A and B on each of the 4 kinds of ruled paper discussed. Find the relation between x and y, the coördinates of any point of the line, if:

1. the given points are $A(2,3)$, $B(4,6)$;
2. " " " " $A(2,3)$, $B(6,4)$;
3. " " " " $A(3,2)$, $B(4,6)$;
4. " " " " $A(3,2)$, $B(6,4)$;
5. " " " " $A(5,3)$, $B(4,6)$.

161. *AP*'s and *GP*'s in a table. If a table of corresponding values of x and y is given in which the x's form an AP whose common difference is c and the y's from an AP whose common difference is d, the graph of the table on S S paper is a straight line whose slope is $\frac{d}{c}$.

The terms of a GP whose ratio is r are a, ar, ar^2, ar^3, ... The logs of these terms are $\log a$, $\log a + \log r$, $\log a + 2 \log r$, $\log a + 3 \log r$, . . ., an AP whose common difference is $\log r$. On the logarithmic scale, the division points are

marked so that the log of the number at a point is the number of unit spaces from the origin to that point. It therefore follows that, for a table such as

$$(1)\quad \begin{array}{c|llll} x & a & a+b & a+2b & a+3b \dots \\ \hline y & c & c+d & c+2d & c+3d \dots \end{array},$$

the graph is a straight line on S S paper;

$$(2)\quad \begin{array}{c|llll} x & a & a+b & a+2b & a+3b \dots \\ \hline y & c & cd & cd^2 & cd^3 \quad \dots \end{array},$$

the graph is a straight line on S L paper;

$$(3)\quad \begin{array}{c|llll} x & a & ab & ab^2 & ab^3 \quad \dots \\ \hline y & c & c+d & c+2d & c+3d \dots \end{array},$$

the graph is a straight line on L S paper;

$$(4)\quad \begin{array}{c|llll} x & a & ab & ab^2 & ab^3 \dots \\ \hline y & c & cd & cd^2 & cd^3 \dots \end{array},$$

the graph is a straight line on L L paper.

The relations between x and y for these forms are

$$(1)\quad \frac{y-c}{x-a} = \frac{d}{b}.$$

$$(2)\quad \frac{\log y - \log c}{x-a} = \frac{\log d}{b}, \text{ or } \frac{y}{c} = d^{(x-a)/b}$$

$$(3)\quad \frac{y-c}{\log x - \log a} = \frac{d}{\log b}, \text{ or } \frac{x}{a} = b^{(y-c)/d}$$

$$(4)\quad \frac{\log y - \log c}{\log x - \log a} = \frac{\log d}{\log b}, \text{ or } \frac{y}{c} = \left(\frac{x}{a}\right)^{\log d/\log b}.$$

In a given table, AP's or GP's may be evident as in:

$$\begin{array}{c|rrrrrrr} x & 3 & 6 & 12 & 24 & 48 & 96 & 192 \\ \hline y & 5 & 8 & 11 & 14 & 17 & 20 & 23 \end{array}.$$

They are not evident in a table such as

$$\begin{array}{c|rrrrr} x & 3 & 12 & 24 & 48 & 192 \\ \hline y & 5 & 11 & 14 & 17 & 23 \end{array},$$

but, if the corresponding numbers 24, 14, are disregarded, the progressions become evident.

Exercise 113

Find the relation between x and y for each of the following tables, and from the relation found calculate the values of p and q.

1.

x	3	9	27	81	p	16
y	2	6	10	14	12	q

2.

x	2	6	10	14	p	7
y	3	9	27	81	50	q

3.

x	2	6	10	14	p	12
y	3	9	15	21	11	q

4.

x	3	9	27	81	p	50
y	1	3	9	27	8	q

5.

x	a	$a+d$	$a+2d$	$a+3d$
y	b	br	br^2	br^3

6.

x	2	5	8	11	14	17	20	p	15
y	3	4.24	6	8.48	12	16.97	24	20	q

Hint: For testing purposes, disregard $x = 5$, 11, 17 and the corresponding y values.

7.

x	2	3.46	6	10.39	18	31.18	p	20
y	1.4	2.1	2.8	3.5	4.2	4.9	3	q

8.

x	2	3	4	6	8
y	.3010	.4771	.6021	.7782	.9031

9.

x	2	4	8	16	32	p	20
y	5	15	45	135	305	50	q

10.

x	2	4	8	16	32	p	20
y	5	10	20	40	80	50	q

11.

x	2	4	8	16	32	p	20
y	405	135	45	15	5	50	q

12.

x	360	180	90	45	22.5	p	60
y	1	2	4	8	16	10	q

13. Given the table of values

x	3	9	27	81
y	2	10	50	250

(a) Find the relation between x and y and check the entire table.

(b) Draw the graph of the relation between x and y on ordinary ruled paper.

(c) Draw the graph of the relation between x and y on log paper.

14. Given the table of values

x	2	6	18	54
y	100	20	4	0.8

(a) Find the relation between x and y and check the entire table.

(b) Draw the graph on log paper.

(c) From the relation between x and y find the value of y if $x = \sqrt{108}$.

15. The coördinates of two points are (1,5) and (3,10) on (a) ordinary ruled paper; (b) semilog paper (y logarithmic); (c) semilog paper (x logarithmic); (d) log paper. If a straight line joins the points, find in each case the relation between x and y, the coördinates of any point on the line.

16. Sketch the graph of $y = 5(2^x)$ on (a) ordinary paper; (b) semilog paper (y logarithmic); (c) semilog paper (x logarithmic).

162. Exponential, parabolic, and hyperbolic curves. The form of relation between x and y, the coördinates of any point of a straight line on semilog paper, is $y = a^x b$, or

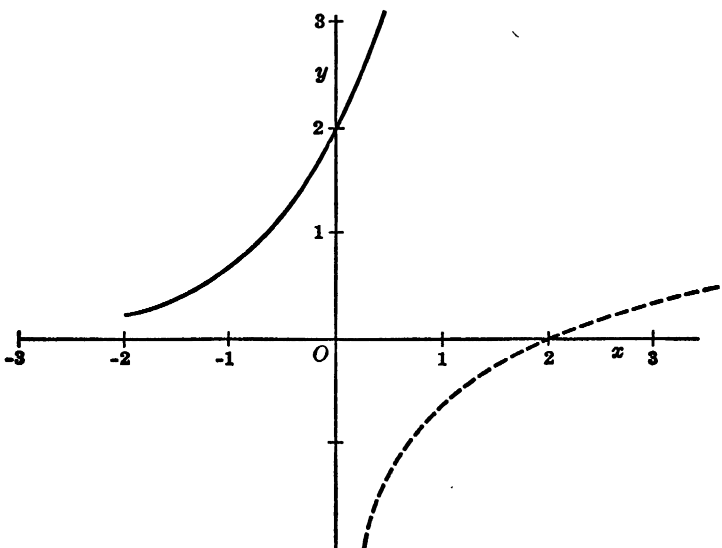


Fig. 41.

$x = a^y b$. The graphs of these equations on ordinary paper are curves called *exponential curves.* The graph of $y = (3^x)2$ in Figure 41 shows that the curve rises rapidly toward the right and does not cross the x axis but approaches it toward the left of the origin. The x axis is an asymptote of the curve. The y axis is an asymptote of the dotted curve, the graph of $x = (3^y)2$.

For a straight line on log paper, the form of relation is $y = x^a b$. On ordinary paper, the graph of this equation is a straight line when $a = 0$ or $a = 1$. For other values of a, the curve is called a *parabola* when a is positive, and a *hyperbola* when a is negative.

The simplest form of the equation of a parabola is obtained when $a = 2$, the form being $y = bx^2$. This curve is symmetrical about the y axis, passes through the origin,

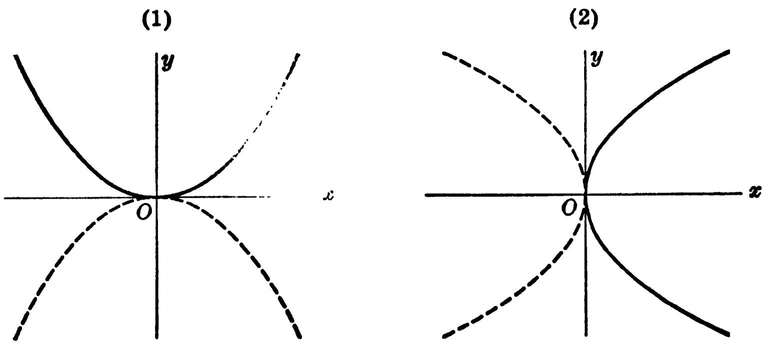


Fig. 42.

and lies above the x axis if b is $+$, and below if b is $-$, as in Fig. 42 (1). The parabola $x = by^2$ is as in Fig. 42 (2).

The simplest form of the equation of the hyperbola is obtained when $a = -1$, the form being $y = bx^{-1}$ or $xy = b$. This curve is symmetrical about the origin and about the 45° line. It consists of two distinct parts for both of which the x and the y axes are asymptotes, as in Fig. 43.

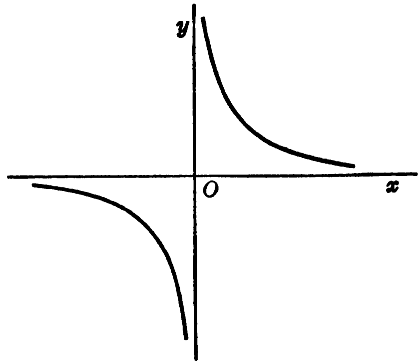


Fig. 43.

Exercise 114

1. Sketch the graphs of the exponential equations on ordinary paper:

(a) $y = 2^x$ (c) $y = 2^{-x}$ (e) $y = -2^x$

(b) $x = 2^y$ (d) $x = 2^{-y}$ (f) $y = (-2)^x$

2. Sketch the parabolas on ordinary paper:

(a) $y = x^2$ (d) $y = x^3$ (g) $x^3 = y^2$

(b) $x = y^2$ (e) $y = -x^3$ (h) $y = x^4$

(c) $y = -x^2$ (f) $y^2 = x^3$

3. Sketch the hyperbolas on ordinary paper:

(a) $y = x^{-1}$ (c) $y = x^{-2}$ (e) $y^2 = x^{-3}$

(b) $x = -y^{-1}$ (d) $y = x^{-3}$ (f) $y^3 = x^{-2}$

4. The graph of some of the equations in examples 1, 2, 3 are straight lines on some kind of ruled paper. State the kind of paper and draw the graph.

5. Which of the equations of examples 1, 2, 3 do not give straight lines on semilog or on log paper?

163. The parabola and the hyperbola. If the coordinate axes are translated by moving the origin to (h, k),

the equation of the parabola $y = mx^2$ becomes $(y' + k) = m(x' + h)^2$, an equation that reduces to the form

$$x^2 + ax + by + c = 0.$$

The parabola $x = ny^2$ takes the form

$$y^2 + ax + by + c = 0.$$

The hyperbola $xy = n$ takes the form

$$xy + ax + by + c = 0.$$

If the axes are now translated by moving the origin to a point on the curve, the curve will pass through the new origin and the constant term, c, will vanish. The three forms will now be

$$\begin{aligned} &(1)\ \ x^2 + ax + by = 0 \\ &(2)\ \ y^2 + ax + by = 0 \\ &(3)\ \ xy + ax + by = 0. \end{aligned}$$

The general form of the equation of a straight line is $AX + BY + C = 0$, a linear equation. The three forms listed above may be transformed to linear equations as follows:

In (1), divide through by x,

$$x + a + b\frac{y}{x} = 0.$$

Now let $X = x$ and $Y = \frac{y}{x}$, and the equation takes the linear form

$$X + bY + a = 0.$$

In (2), divide through by y,

$$y + a\frac{x}{y} + b = 0.$$

Now let $X = \frac{x}{y}$ and $Y = y$, and the equation takes the linear form

$$Y + aX + b = 0.$$

In (3), divide through by y,

$$x + a\frac{x}{y} + b = 0.$$

Now let $X = x$ and $Y = \frac{x}{y}$, and the equation takes the linear form

$$X + aY + b = 0.$$

In (3), divide through by x,

$$y + a + b\frac{y}{x} = 0.$$

Now let $X = \frac{y}{x}$ and $Y = y$, and the equation takes the linear form

$$Y + bX + a = 0.$$

If a table of corresponding values of x and y is given and the graph is not a straight line on ordinary, on semilog, or on log paper, it may be a parabola or a hyperbola on ordinary paper. Whether it is or is not a parabola or a hyberbola is determined as follows:

Illustration 1

x		y	(1) x'	(2) y'	(3) x'/y'	(4) y'/x'
-4		16	-2	12	$-\frac{1}{6}$	-6
-3		7	-1	3	$-\frac{1}{3}$	-3
-2	$\longleftrightarrow$	4	0	0	$\frac{0}{0}$	$\frac{0}{0}$
-1		2.5	1	$-\frac{3}{2}$	$-\frac{2}{3}$	$-\frac{3}{2}$
1		1	3	-3	-1	-1
4		0	6	-4	$-\frac{3}{2}$	$-\frac{2}{3}$
13		-1	15	-5	-3	$-\frac{1}{3}$

The table x, y was given. In the tabulation, the origin was moved to $(-2, 4)$, though any other point given by the x, y table would do as well. Since $x = -2 + x'$ and $y = +4 + y'$, columns (1) and (2) are

written by merely subtracting -2 from each x and $+4$ from each y. Column (3) is formed by dividing x' by y', and column (4) is the reciprocal of column (3).

All that is now necessary is to determine, by examining the slopes, whether the pair of columns (1) – (3), (1) – (4), (2) – (3), or (2) – (4) gives a straight line graph. For this purpose, disregard the terms $\frac{0}{0}$. In this case, columns (1) – (3) show the constant slope $-\frac{1}{6}$. Therefore let $X = x'$ and $Y = \frac{x'}{y'}$, and the relation is

$$6Y + X = -3, \quad \text{or } \frac{6x'}{y'} + x' = -3, \quad \text{or } 6x' + x'y' + 3y' = 0.$$

Finally,

$$6(x + 2) + (x + 2)(y - 4) + 3(y - 4) = 0, \text{ or}$$
$$xy + 2x + 5y - 8 = 0.$$

Illustration 2

x		y	(1) x'	(2) y'	(3) x'/y'	(4) y'/x'
22		6	12	-24	$-\frac{1}{2}$	-2
19		9	9	-21	$-\frac{3}{7}$	$-\frac{7}{3}$
16		14	6	-16	$-\frac{3}{8}$	$-\frac{8}{3}$
13		21	3	-9	$-\frac{1}{3}$	-3
10	$\longleftrightarrow$	30	0	0	$\frac{0}{0}$	$\frac{0}{0}$
7		41	-3	11	$-\frac{3}{11}$	$-\frac{11}{3}$
4		54	-6	24	$-\frac{1}{4}$	-4
1		69	-9	39	$-\frac{3}{13}$	$-\frac{13}{3}$

The origin was moved to (10, 30), the relations being $x = x' + 10$ and $y = y' + 30$. Columns (1) – (4) show the constant slope $\frac{1}{9}$. Hence if $Y = y'/x'$, and $X = x'$,

$$9Y - X = -30, \quad \text{or } \frac{9y'}{x'} - x' + 30 = 0, \quad \text{or } 9y' - x'^2 + 30x' = 0.$$

The relation between x and y is

$$9(y - 30) - (x - 10)^2 + 30(x - 10) = 0, \text{ or } x^2 - 50x - 9y + 670 = 0.$$

Exercise 115

Find the relation between x and y for each of the following tables:

1.

x	2	3	4	7	10
y	17	8	5	2	1

2.

x	0	1	2	3	4	5
y	2	1	4	11	22	37

3.

x	5	6	3	-4	-15	-30
y	0	1	2	3	4	5

4.

x	3	4	5	8	11
y	21	12	9	6	5

5.

x	1	2	3	5	8
y	10	7	5.5	4	3

6.

x	2	1	4	11	22
y	0	1	2	3	4

7.

x	1	2	3	4	5
y	2	8	20	38	62

8.

x	16	7	4	1	-1
y	4	5	6	9	21

9.

x	3	4	5	6	8	11	20
y	21	12	9	7.5	6	5	4

164. Parametric equations. From a given equation, it may be very difficult to make up a table of corresponding values of x and y, and to draw the graph. Thus the equation $x^3 + y^3 = 6xy$ shows that the graph is symmetrical about the 45° line and that it passes through the origin, but, if $x = 1$, y must be found from $y^3 - 6y + 1 = 0$, a difficult problem. Such difficulties may be avoided by expressing x and y in terms of a third variable, t.

In this case, if we write $y = tx$, $x^3 + y^3 = 6xy$ becomes $x^3 + t^3x^3 = 6tx^2$, and we obtain the two equations

$$x = \frac{6t}{1+t^3}, \qquad y = \frac{6t^2}{1+t^3},$$

called *parametric equations.* The variable t is called a parameter, something by means of which x and y are measured.

A table of values of x and y can now be made by assigning to t the values $\ldots, -3, -2\frac{1}{2}, -2, -1\frac{1}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, \ldots.$

Conversely, the relation between x and y for a given table may often be found by adding a third column, t. Then if relations can be found between x and t, and also between

y and t, the parametric equations thus found constitute the relation between x and y.

Illustration

x	4	9	16	25	36	49
y	3	6	12	24	48	96
t	2	3	4	5	6	7

The values of t added to the given table form an AP.
The relation between x and t is $x = t^2$.
The relation between y and t is $y = 3(2^{t-2})$.
If we so desire, we may eliminate t from the parametric equations.

Exercise 116

Find the relation between x and y for the following tables by introducing a parameter t.

1.

x	3	6	12	24	48	96
y	1	8	27	64	125	216

2.

x	3	6	12	24	48	96
y	7	11	17	25	35	47

3.

x	7	11	17	35	25	47
y	5	8	14	23	35	50

4.

x	3	6	12	24	48	96
y	5	15	45	135	405	1215

5.

x	9	16	25	36	49	64
y	2	6	18	54	162	486

165. Summary. To discover the relation between x and y for a given table, the following procedure is suggested:

(1) The relation may be obvious.

(2) If the slope is constant, the form is $ax + by + c = 0$.

(3) If the terms of one of the variables form an AP and the corresponding values of the other variable form a GP, the form is $y = a^x b$, or $x = a^y b$. If the progressions are not evident, plotting on semilog paper may furnish a clew.

(4) If both variables form GP's, the form is $y = x^a b$. Plotting on log paper may furnish a clew in this case.

(5). If the foregoing relations are not present, the relation may be one of the forms

$$x^2 + ax + by + c = 0, \quad y^2 + ax + by + c = 0, \text{ or}$$
$$xy + ax + by + c = 0.$$

(6). Introduce a third variable t and see whether parametric equations can be made.

Note: From a given equation, a table can be constructed and a graph drawn. When a table is given, however, many different equations are possible. Thus for the table

x	1	2	3	4
y	1	4	9	16,

one form of relation is $y = x^2$, a parabola. But the equation

$$y = x^2 + (x-1)(x-2)(x-3)(x-4)$$

also gives a graph which passes through the points represented by the given table. In general, the equation for this table may be written

$$y = x^2 + (x-1)(x-2)(x-3)(x-4)P,$$

where P may be practically any algebraic expression containing x. It is therefore to be understood that, when a table is given, it is the equation of simplest form that is required.

Exercise 117

1. Form a table of 6 corresponding values of x and y from the following equations:

(a) $y = 3x - 5$

(b) $y = 2 \cdot x^3$

(c) $y = 2 \cdot 3^x$

(d) $x = 3 \cdot 2^y$

(e) $y^2 = x + 2y - 3$

(f) $x^2 = 2x - y + 3$

(g) $xy = 3x + y - 2$

(h) $y = 4 \cdot 3^{\sqrt{x}}$

2. From the tables formed in example 1, find the equations between x and y.

3. Find the relation between x and y for each of the following tables:

(a)

x	1	4	9	16	25
y	3	8	13	18	23

(b)

x	3	5	7	9	11
y	2	10	50	250	1250

(c)

x	2	6	18	54	162
y	3	6	9	12	15

(d)

x	5	10	20	40	80
y	4	12	36	108	324

(e)

x	0	1	2	3	4
y	11	21	35	53	75

(f)

x	2	6	18	54	162
y	1	8	27	64	125

CHAPTER XIV

LEAST SQUARE SOLUTION

166. Best relation. It is not to be expected that an exact relation exists between x and y, corresponding values of two variables in a table of statistical data, since y usually depends, not on x alone, but on many other, generally unknown, variables as well. The values of x and y are, however, associated or correlated, and the statistician is faced with two problems: (1) What is the best algebraic relation between x and y? (2) How is the degree of correlation to be measured?

Each pair of corresponding values locates a point. If the aggregate of the points, called a *scatter diagram*, suggests a curve, the equation of the curve is the desired relation.

A polynomial in x and y which contains only positive integral powers of x and y is called an *algebraic polynomial;* x and y may not appear as exponents or as denominators of fractions. The coefficients in the separate terms are constants which may be any real numbers. Thus

$$x^3 - \tfrac{2}{3}x^2y + 5x^2 - \tfrac{5}{9}xy + 4$$

is an algebraic polynomial.

Suppose a table of corresponding values of x and y is given and the type of relation between x and y selected as a possible form to fit the table is $P = 0$, where P is an algebraic polynomial. The coefficients in the various terms are unknown constants, and the number of these coefficients is less than the number of terms in P by 1, since $P = 0$ can be divided by any one of the coefficients.

One set of values of these coefficients will make $P = 0$ fit the table better than another set, and it is our object to find such a set of values of the coefficients.

The substitution of x_1, y_1 for x, y in P gives P_1, which may be $+$, $-$, or 0, and similarly for P_2, P_3, and so on. Now, if $P_1 + P_2 + \ldots + P_n = \Sigma P$ is zero, the meaning is that negative values of P are offset by positive values, and the divergences from 0 may be large. But if P_1, P_2, and so forth are squared, $\Sigma P^2 = P_1^2 + P_2^2 + \ldots + P_n^2$ can be 0 only if $P_1 = P_2 = \ldots = P_n = 0$. Such a result would mean that each pair of corresponding values in the table fits the equation $P = 0$, and therefore the exact relation $P = 0$ fits the table.

The nearer to 0 that ΣP^2 is, the better will $P = 0$ fit the table. Therefore, the best fit for a curve of the type selected is the one obtained so as to make ΣP^2 least. The equation $P = 0$ in which the values of the coefficients are such as to make ΣP^2 least is called a *least square solution.*

Since the coefficients in $P = 0$ are all of the first degree, ΣP^2 is a quadratic in each of them, and a least square solution is one for which a quadratic trinomial of the form $At^2 + Bt + C$ has a minimum value.

167. The quadratic $P = At^2 + Bt + C$. Any quadratic trinomial such as $\frac{1}{2}t^2 - 2t + 5$ has a value P for each different value of t. But $\frac{1}{2}t^2 - 2t + 5 - P = 0$ when solved for t gives $t = 2 \pm \sqrt{2(P - 3)}$. Hence in order that t shall be a real number, P may not be less than 3. That is, the minimum value of P or of $\frac{1}{2}t^2 - 2t + 5$ is 3 and this value of P arises when $t = 2 \pm \sqrt{2(3 - 3)} = 2$.

In general, if $At^2 + Bt + C = P$, then

$$t = \frac{-B \pm \sqrt{4AP + B^2 - 4AC}}{2A} = \frac{-B \pm \sqrt{4A\left(P - \dfrac{4AC - B^2}{4A}\right)}}{2A}$$

Now if A is + and t is to be real, P may not be less than

$$\frac{4AC - B^2}{4\,A}$$

and P has this minimum value when $t = -\frac{B}{2A}$. That is, $At^2 + Bt + C$ is a minimum when A is + and $t = -\frac{B}{2A}$.

Illustrations

1. Given $P = 2x^2 - 3xy + 4y^2 - x - 5y + 6$.

In this quadratic in x, $A = +2$, $B = -3y - 1$, $C = 4y^2 - 5y + 6$. Hence one condition for P to be a minimum is that

$$x = -\frac{B}{2A} = \frac{3y + 1}{4}.$$

P is also a quadratic in y, and

$$A = +4,$$
$$B = -3x - 5.$$

Hence another condition for P to be a minimum is that

$$y = \frac{3x + 5}{8}.$$

The values of x and y for which P is a minimum are found from the two conditional or simultaneous equations

$$x = \frac{3y + 1}{4},$$

and

$$y = \frac{3x + 5}{8}.$$

The values are $x = 1$, $y = 1$, and the minimum value of P is $+3$.

2. If there are n corresponding values of x and y in a table, find the values of a and b for which

$$M = \Sigma\,(a + bx - y)^2$$

is a minimum.

Expanded with reference to a,

$$M = \Sigma\,[a + (bx - y)]^2 = \Sigma\,[a^2 + 2a\,(bx - y) + (bx - y)^2]$$
$$= na^2 + 2a\Sigma\,(bx - y) + \Sigma\,(bx - y)^2.$$

One condition for M to be a minimum is, therefore,

$$a = \frac{-2\Sigma\,(bx - y)}{2\,n}, \text{ or } an + b\Sigma x = \Sigma y.$$

M may also be expanded with reference to b, giving

$$M = \Sigma\,[bx + (a - y)]^2 = b^2\Sigma\, x^2 + 2b\Sigma\, x\,(a - y) + \Sigma\,(a - y)^2.$$

Hence another condition for M to be a minimum is

$$b = \frac{-2\Sigma x\,(a - y)}{2\Sigma x^2}, \text{ or } b\Sigma x^2 + a\Sigma x = \Sigma xy.$$

The values of a and b for which M is a minimum is found from the simultaneous equations

$$\begin{cases} an + b\Sigma x = \Sigma y \\ a\Sigma x + b\Sigma x^2 = \Sigma xy. \end{cases}$$

Thus the following table in x and y is given, and the columns for x^2, xy, y^2 are added.

x	y	x^2	xy	y^2
2	1	4	2	1
3	4	9	12	16
4	7	16	28	49
5	10	25	50	100
6	13	36	78	169
20	35	90	170	335

$n = 5$ (five items), $\Sigma x = 20$, $\Sigma y = 35$, $\Sigma x^2 = 90$, $\Sigma xy = 170$, $\Sigma y^2 = 335$.

The polynomial $M = \Sigma(a + bx - y)^2$ is a minimum for the given table if $a = -5$ and $b = 3$, found from the simultaneous equations

$$5a + 20b = 35$$

$$20a + 90b = 170.$$

For these values of a and b, $a + bx - y$ becomes $-5 + 3x - y$.

$$M = \Sigma(-5 + 3x - y)^2 = \Sigma(25 + 9x^2 + y^2 - 30x + 10y - 6xy)$$
$$= 5 \times 25 + 9 \times 90 + 335 - 30 \times 20 + 10 \times 35 - 6 \times 170 = 0.$$

The student may show that, for any other values of a and b, $M > 0$.

Exercise 118

1. State the values of x for which P is a minimum if:

(a) $P = 2x^2 - 5x + 7$ (b) $P = 3x^2 + 9x - 10$

2. Show that $P = x^2 - 2xy + 3y^2 - 5x + y + 12$ is a minimum when $x = 3\frac{1}{2}$, $y = 1$, and that the minimum value of P is $\frac{15}{4}$.

3. Find the value of P in example 2 if:

(a) $x = 3, y = 1\frac{1}{2}$ (c) $x = 4, y = \frac{1}{2}$
(b) $x = 3, y = \frac{1}{2}$ (d) $x = 4, y = 2$

4. Show that the following do not have minimum values:

(a) $7 - 5x - x^2$ (b) $5 + 8x - 3x^2$
(c) $x^2 - 3xy - y^2 + 2x + 4y + 5$

5. Given the table of corresponding values

x	1	3	5	7	10	12
y	3	4	8	11	13	14

(a) Find the value of a for which M is a minimum if:

(1) $M = \Sigma(ax - y)^2$ (2) $M = \Sigma(x - ay)^2$

(b) Find the values of a and b for which M is a minimum if $M = \Sigma(ax + by + 1)^2$.

168. Normal equations. The constant coefficients a, b, c, and so on for a least square solution are determined by setting up as many equations, called *normal equations*, as there are coefficients. The method of forming the normal equations is perfectly general and is applicable to any form of equation selected.

Suppose the form selected is

$$P = xy + ax + by + c = 0,$$

and

$$\Sigma P^2 = \Sigma(xy + ax + by + c)^2$$

is to be a minimum. To expand ΣP^2 with reference to a, write it as

$$\Sigma(ax + k)^2,$$

where

$$k = xy + by + c.$$

Then

$$\Sigma P^2 = \Sigma(ax + k)^2 = a^2\Sigma x^2 + 2a\Sigma kx + \Sigma k^2.$$

This quadratic in a is a minimum when

$$a = \frac{-2\Sigma kx}{2\Sigma x^2},$$

or when

$$a\Sigma x^2 + \Sigma kx = 0.$$

The last equation may be written

$$\Sigma(ax^2 + kx) = \Sigma x(ax + k) = \Sigma x(xy + ax + by + c) = \Sigma xP = 0.$$

That is, one condition for ΣP^2 to be a minimum is the equation $\Sigma xP = 0$, where x is the multiplier of a in the polynomial P.

Similarly, if ΣP^2 is expanded with reference to b, the multiplier of which is y, the condition for ΣP^2 to be a minimum is $\Sigma yP = 0$.

If ΣP^2 is expanded with reference to c, the multiplier of which is 1, the condition is $\Sigma 1P = 0$.

The three normal equations are

$$\Sigma xP = \Sigma yP = \Sigma 1P = 0,$$

or, when expanded,

$$\begin{aligned} &\Sigma x^2y + a\Sigma x^2 + b\Sigma xy + c\Sigma x = 0, \\ &\Sigma xy^2 + a\Sigma xy + b\Sigma y^2 + c\Sigma y = 0, \\ &\Sigma xy + a\Sigma x + b\Sigma y + nc = 0. \end{aligned}$$

Exercise 119

Write the normal equations from which the coefficients a, b, c, and so on are to be determined for a least square solution if the form of equation selected is:

1. $P = x^2 + ax + by + c = 0$
2. $P = y^2 + ax + by + c = 0$
3. $P = -y + a + bx + cx^2 + dx^3 = 0$
4. $P = -x + a + by + cy^2 + dy^3 = 0$

5. $P = -y + a + bx = 0$

6. $P = -x + a + by = 0$

7. $P = -y + ax = 0$

8. $P = -x + ay = 0$

9. $P = 1 + axy + bx + cy = 0$

10. $P = x^2y + ax^2 + bxy + cy^2 + dx + ey + f = 0$

169. Relative closeness of fit. How well the type of equation selected fits the given table of values is shown by the value of ΣP^2. If ΣP^2 is small, the fit is good, but it is difficult to tell what number shall be considered small. However, if two different types of equation are selected, that type is the better fit for which ΣP^2 is smaller. Therefore a simple method of calculating ΣP^2 is desirable.

Suppose the hyperbola $P = xy + ax + by + c = 0$ is selected.

$$\Sigma P^2 = \Sigma P\,(xy + ax + by + c) = \Sigma xyP + a\Sigma xP + b\Sigma yP + c\Sigma 1P.$$

But the normal equations from which a, b, c are determined are

$$\Sigma xP = 0, \quad \Sigma yP = 0, \quad \Sigma 1P = 0.$$

Therefore ΣP^2 reduces to ΣxyP, and

$$\Sigma P^2 = \Sigma xy\,(xy + ax + by + c) = \Sigma x^2y^2 + a\Sigma x^2y + b\Sigma xy^2 + c\Sigma xy.$$

In general, if P is any algebraic polynomial and $P = 0$ is the type of equation selected, $\Sigma P^2 = \Sigma$ (P multiplied by that term in P which does not contain any of the coefficients). Thus for the form

$$P = a + bx + cx^2 + dx^3 - y = 0,$$

$$\Sigma P^2 = \Sigma\,(-y)P = \Sigma y^2 - \Sigma y\,(a + bx + cx^2 + dx^3).$$

Exercise 120

Write the formula for ΣP^2 for each of the equations of Exercise 119, page 286.

170. Geometric meaning of a least square solution. The usual type of equation selected is one in which y is expressed as a polynomial in x, or one in which x is expressed as a polynomial in y. The usual types are

$$y = a + bx + cx^2 + dx^3 + \ldots,$$

or

$$P = -y + a + bx + cx^2 + dx^3 + \ldots = 0,$$

and

$$x = a + by + cy^2 + dy^3 + \ldots,$$

or

$$Q = -x + a + by + cy^2 + dy^3 + \ldots = 0,$$

called *parabolas of the 1st, 2nd, 3rd order*, according as the highest power of x in P or of y in Q is 1, 2, or 3, the parabola of the first order being a straight line.

An equation of the type $P = 0$ means that we assume the x values of the given table to be correct and from them

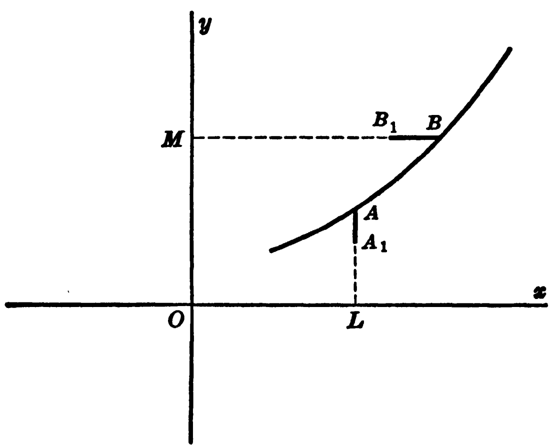


Fig. 44.

we calculate the y values. An equation of the type $Q = 0$ means that we assume the y values of the given table to

be correct and from them we calculate the x values. For these types, least square solutions have simple geometric meanings.

If the equation is type $P = 0$, point A_1, Fig 44, represents the corresponding values $x_1 = OL$, $y_1 = LA_1$. In the equation $P = 0$, the substitution of OL for x gives $y = LA$. The deviation of LA_1 measured vertically from the curve, called the *vertical deviation of* A_1 is $P_1 = LA - LA_1 = AA_1$. The least square solution is the result of the conditions that the sum of the squares of the vertical deviations, ΣP^2, shall be a minimum.

If the equation is type $Q = 0$, point B_1 represents the corresponding values $y_1 = OM$, $x_1 = MB_1$. In the equation $Q = 0$, the substitution of OM for y gives $x = MB$. The *horizontal deviation* of B_1 from the curve is $MB - MB_1 = BB_1$, and the least square solution is the result of the conditions that the sum of the squares of the horizontal deviations, ΣQ^2, shall be a minimum.

171. Simplification of calculations. The normal equations for

$$P = -y + a + bx + cx^2 + dx^3 = 0$$

are

$$\Sigma 1P = 0,\ \Sigma Px = 0,\ \Sigma Px^2 = 0,\ \Sigma Px^3 = 0.$$

When these equations are expanded, there are terms containing Σx, Σx^2, Σx^3, Σx^4, Σx^5, Σx^6, Σy, Σxy, Σx^2y, Σx^3y, and the solution of the simultaneous equations requires a considerable amount of arithmetic calculation.

If, however, the x's in the given table form an AP, say $x = 1, 3, 5, 7, 9, 11, 13$, the value of $\bar{x}$ is 7. Move the origin to (7, 0) so that $x' = x - 7 = -6, -4, -2, 0, 2, 4, 6$. It is now evident that

$$\Sigma x' = 0,\ \Sigma x'^3 = 0,\ \Sigma x'^5 = 0,$$

but $\Sigma x'y'$ and $\Sigma x'^3y'$ are not 0. For such a case, the normal equations reduce to

$$\begin{aligned} na + c\Sigma x'^2 &= \Sigma y \\ b\Sigma x'^2 + d\Sigma x'^4 &= \Sigma x'y \\ a\Sigma x'^2 + c\Sigma x'^4 &= \Sigma x'^2y \\ b\Sigma x'^4 + d\Sigma x'^6 &= \Sigma x'^3y. \end{aligned}$$

Now a and c are easily found from the 1st and 3rd equations, and b and d from the 2nd and 4th equations. To test the closeness of fit,

$$\Sigma P^2 = \Sigma(-y)P = \Sigma(-y)(-y + a + bx' + cx'^2 + \ldots).$$

Illustration

Given the table

x	2	3	4	5	6	7	8
y	5	6	9	10	12	10	8

It is required to find the best parabola to fit the table. The x's form an AP and $\bar{x} = 5$. The calculations are simplified by moving the origin to $(5, k)$, where k may be 0 or any other number. In the following tabulation, k is taken as 10, the y value corresponding to $x = 5$, so that $x' = x - 5$ and $y' = y - 10$.

x	y	x'	y'	x'^2	x'^4	x'^6	$x'y'$	x'^2y'	x'^3y'	y'^2
2	5	−3	−5	9	81	729	15	−45	135	25
3	6	−2	−4	4	16	64	8	−16	32	16
4	9	−1	−1	1	1	1	1	− 1	1	1
5←	→10	0	0	0	0	0	0	0	0	0
6	12	1	2	1	1	1	2	2	2	4
7	10	2	0	4	16	64	0	0	0	0
8	8	3	−2	9	81	729	−6	−18	−54	4
35	60	0	−10	28	196	1588	20	−78	116	50

The normal equations for the form $y' = a + bx'$ are $7a = -10$ and $28b = 20$, or $a = -\frac{10}{7}$, $b = \frac{5}{7}$.

The normal equations for the form $y' = a + bx' + cx'^2$ are $7a + 28c = -10$, $28b = 20$, $28a + 196c = -78$, or $a = \frac{8}{21}$, $b = \frac{5}{7}$, $c = -\frac{19}{42}$.

The normal equations for the form $y' = a + bx' + cx'^2 + dx'^3$ are

$7a + 28c = -10$, $28b + 196d = 20$, $28a + 196c = -78$, $196b + 1588d = 116$, or $a = \frac{8}{21}$, $b = \frac{94}{63}$, $c = -\frac{19}{42}$, $d = -\frac{1}{9}$.

Each of the least square solutions,

$$y' = -\tfrac{10}{7} + \tfrac{5}{7}x', \ \Sigma P^2 = 21\tfrac{3}{7},$$
$$y' = \tfrac{8}{21} + \tfrac{5}{7}x' - \tfrac{19}{42}x'^2, \ \Sigma P^2 = 4\tfrac{5}{21},$$
$$y' = \tfrac{8}{21} + \tfrac{94}{63}x' - \tfrac{19}{42}x'^2 - \tfrac{1}{9}x'^3, \ \Sigma P^2 = 1\tfrac{4}{7},$$

is a "best" fit for the table of values. The parabola of the second order is much better than the line, and the parabola of the third order is an exceedingly close fit.

Exercise 121

1. In the illustration, (a) verify the normal equations; (b) verify the solutions of the normal equations; (c) verify the values of ΣP^2.

2. From the equations in x', y', find the equations in x, y.

3. Draw the graphs of the three least square solutions, plotting the points for $x' = -3, -2, -1, 0, 1, 2, 3$.

4. Show that ΣP^2 remains unchanged whether the axes are translated or not.

5. Move the origin to (5, 9) instead of to (5, 10) and obtain the values of the coefficients and of ΣP^2 for each of the three forms of equation.

6. Given the table

x	2	3	4	5	6	7	8	9	10
y	3	5	8	12	10	7	8	12	15,

move the origin to (6, 10) and find:

(a) the least square solution of the form $P = -y' + a + bx' = 0$ and ΣP^2.

(b) the least square solution of the form $P = -y' + a + bx' + cx'^2 = 0$ and ΣP^2.

(c) the least square solution of the form $P = -y' + a + bx' + cx'^2 + dx'^3 - 0$ and ΣP^2.

172. Regression lines. The equation of a curve may sometimes be expressed in the form y = a polynomial in x and sometimes in the form x = a polynomial in y, but if one of the forms is possible the other is generally not

possible. Therefore, if, for an assumed form of curve, a least square solution can be obtained so that the sum of the squares of the vertical deviations is a minimum, it is generally not possible to find a least square solution so that the sum of the squares of the horizontal deviations is a minimum.

A linear equation, however, whose general form is $Ax + By + C = 0$ can be written in the form

$$y = a_1 + b_1x, \text{ or } P = -y + a_1 + b_1x = 0,$$

or in the form

$$x = a_2 + b_2y, \text{ or } Q = -x + a_2 + b_2y = 0.$$

Therefore, two different linear least square solutions are possible for a given table of values of x and y, both of which have geometric meanings. The two solutions are called *regression equations* and the corresponding lines are called *regression lines*. The values of ΣP^2 and ΣQ^2 are different, and, although one of them may be nearer zero than the other, one solution is as good as the other since they are least square solutions in different senses. There are, therefore, two "best" lines.

Illustration

In the following table, x and y are given, and the columns x^2, xy, y^2 are added:

x	y	x^2	xy	y^2
2	5	4	10	25
3	6	9	18	36
4	9	16	36	81
5	10	25	50	100
6	12	36	72	144
7	10	49	70	100
8	8	64	64	64
35	60	203	320	550

$$\Sigma x = 35,\ \Sigma y = 60,\ \Sigma x^2 = 203,\ \Sigma xy = 320,\ \Sigma y^2 = 550,\ n = 7.$$

For the form $P = -y + a_1 + b_1x = 0$, the normal equations are $\Sigma 1P = 0, \Sigma xP = 0$, or $na_1 + b_1\Sigma x = \Sigma y$, $a_1\Sigma x + b_1\Sigma x^2 = \Sigma xy$, or $7a_1 + 35b_1 = 60$, $35a_1 + 203b_1 = 320$, and $a_1 = 5$, $b_1 = \frac{5}{7}$. That is, the equation of one regression line is $y = 5 + \frac{5}{7}x$.

For this solution, $\Sigma P^2 = \frac{150}{7} = 21\frac{3}{7}$.

For the form $Q = -x + a_2 + b_2y = 0$, the normal equations are $na_2 + b_2\Sigma y = \Sigma x$, $a_2\Sigma y + b_2\Sigma y^2 = \Sigma xy$, from which $a_2 = \frac{1}{5}$, $b_2 = \frac{14}{25}$.

For this solution, $\Sigma Q^2 = \frac{84}{5} = 16\frac{4}{5}$, the equation of the regression line being $x = \frac{1}{5} + \frac{14}{25}y$.

The two regression lines are different since the second line reduces to $y = \frac{25}{14}x - \frac{5}{14}$, whereas the first is $y = \frac{5}{7}x + 5$. They intersect at $x = 5$, $y = 8\frac{4}{7}$, the point whose coördinates are $\bar{x}$ and $\bar{y}$.

Exercise 122

Find the equations of the regression lines, the sum of the squares of the deviations, and the intersection point, for each of the tables:

1.

x	1	3	5	7	9
y	2	5	9	11	13

2.

x	1	3	5	7	9
y	2	5	8	11	14

3.

x	1	3	5	6	7
y	1	5	6	9	10

4.

x	1	2	3	4	5
y	11	7	5	4	1

5.

x	5	7	9	11	13
y	20	14	11	6	4

173. Intersection of regression lines. A table of corresponding values of x and y having been represented by points, let the origin be moved to $(\bar{x}, \bar{y})$ by a translation of the axes, and let the new coördinates be designated by (x', y'). Then $x' = x - \bar{x}$ and $y' = y - \bar{y}$, are the deviations from the means of the x's and y's, respectively.

For the regression line $y' = a_1 + b_1x'$, the normal equations are $\Sigma y' = na_1 + b_1\Sigma x'$ and $\Sigma x'y' = a_1\Sigma x' + b_1\Sigma x'^2$. But since $\Sigma x' = \Sigma(x - \bar{x}) = 0$, and $\Sigma y' = \Sigma(y - \bar{y}) = 0$, the normal equations give $a_1 = 0$ and $b_1 = \dfrac{\Sigma x'y'}{\Sigma x'^2}$. The equation of the regression line is $y' = b_1x'$, a line that passes through the new origin.

Similarly, the normal equations for the regression line $x' = a_2 + b_2y'$, are $\Sigma x' = na_2 + b_2\Sigma y'$, and $\Sigma x'y' = a_2\Sigma y' + b_2\Sigma y'^2$, which give $a_2 = 0$ and $b_2 = \frac{\Sigma x'y'}{\Sigma y'^2}$. The equation of this regression line is $x' = b_2y'$, a line which also passes through the new origin.

Therefore the regression lines intersect at the point $(\bar{x}, \bar{y})$, and their equations are

$$y' = b_1x' = \frac{\Sigma x'y'}{\Sigma x'^2}x'$$

and

$$x' = b_2y' = \frac{\Sigma x'y'}{\Sigma y'^2}\, y'.$$

If the average product of the deviations from the mean, $\frac{\Sigma x'y'}{n}$, is represented by p, then, since

$$\Sigma x'^2 = \Sigma(x - \bar{x})^2 = n\sigma_x^2,$$

and

$$\Sigma y'^2 = \Sigma(y - \bar{y})^2 = n\sigma_y^2,$$

the equations of the regression lines may be written

$$y' = \frac{p}{\sigma_x^2}\, x', \text{ and } x' = \frac{p}{\sigma_y^2}\, y'.$$

174. Coefficient of correlation. If the points that represent a given table are exactly on a line—that is, if there is an exact linear relation between x and y—the two regression lines coincide and

$$\frac{p}{\sigma_x^2} = \frac{\sigma_y^2}{p} \text{ or } \frac{p^2}{\sigma_x^2\sigma_y^2} = 1.$$

If there is not an exact linear relation, $\frac{p^2}{\sigma_x^2\sigma_y^2}$ differs from 1. The fraction $\frac{p^2}{\sigma_x^2\,\sigma_y^2}$, which is equal to b_1b_2, is represented by r^2, so that

$$r = \pm\frac{p}{\sigma_x\sigma_y},$$

and r is called the *coefficient of linear correlation.* The value of r indicates whether or not a close linear relation exists between x and y. Since $r^2 = b_1b_2$, if $r = 0$, either $b_1 = 0$ or $b_2 = 0$. In either case, x and y are not related at all. If $r^2 = 1$, there is a perfect linear relation between x and y. For the form $P = -y' + b_1x' = 0$,

$$\Sigma P^2 = \Sigma(-y')(-y' + b_1x') = \Sigma y'^2 - b_1\Sigma x'y' = n\sigma_y^2 - \frac{p}{\sigma_x^2} \cdot np$$

$$\Sigma P^2 = n\sigma_y^2\left(1 - \frac{p^2}{\sigma_x^2\sigma_y^2}\right) = n\sigma_y^2(1 - r^2).$$

Similarly, for the form $Q = -x' + b_2y' = 0$,

$$\Sigma Q^2 = n\sigma_x^2\,(1 - r^2).$$

Now, since ΣP^2 and ΣQ^2 cannot be negative, r^2 cannot be greater than 1, and r is between -1 and $+1$. Negative values of r arise when y increases while x decreases, or when y decreases while x increases. The values of σ_x and σ_y are always taken as positive so that r is negative when p or $\Sigma x'y'/n$ is negative. Thus for the table on page 292, $b_1 = \frac{5}{7}$ and $b_2 = \frac{14}{25}$. Hence

$$r^2 = b_1b_2 = \frac{5}{7} \times \frac{14}{25} = \frac{2}{5} = .40, \text{ and } r = .63.$$

This result indicates that a linear relation between x and y is fairly good but not very close.

175. Calculation of r. When $\bar{x}$ and $\bar{y}$ are not integers, the value of

$$r^2 = \frac{p^2}{\sigma_x^2\sigma_y^2} = \frac{[\Sigma(x - \bar{x})\,(y - \bar{y})]^2}{\Sigma(x - \bar{x})^2\,\Sigma(y - \bar{y})^2}$$

is calculated by following the method used for calculating $\bar{x}$, page 237. Move the origin to an arbitrary point (h, k) near the mean point by a translation of the axes and let $\bar{x} = h + d_1$ and $\bar{y} = k + d_2$. Then

$$\Sigma(x - h) = \Sigma(x - \bar{x} + d_1) = nd_1;$$
$$\Sigma(y - k) = \Sigma(y - \bar{y} + d_2) = nd_2.$$

$$\begin{aligned}
\Sigma(x-h)^2 &= \Sigma(x-\bar{x})^2 + 2d_1\Sigma(x-\bar{x}) + \Sigma d_1^2 = n\sigma_x^2 + nd_1^2 \\
\Sigma(y-k)^2 &= \Sigma(y-\bar{y})^2 + 2d_2\Sigma(y-\bar{y}) + \Sigma d_2^2 = n\sigma_y^2 + nd_2^2 \\
\Sigma(x-h)(y-k) &= \Sigma(x-\bar{x}+d_1)(y-\bar{y}+d_2) \\
&= \Sigma(x-\bar{x})(y-\bar{y}) + \Sigma d_1 d_2 \\
&= np + nd_1 d_2.
\end{aligned}$$

Since the tabulation gives the values of $\Sigma(x-h)$, $\Sigma(y-k)$, $\Sigma(x-h)(y-k)$, $\Sigma(x-h)^2$, $\Sigma(y-k)^2$, there is now no difficulty in calculating σ_x^2, σ_y^2, and p.

Illustration

The grades in percentages are given for 20 students in history, x, and in economics, y, as in the table. $\Sigma x = 1665$ and $\Sigma y = 1618$, giving $\bar{x} = 82.25$ and $\bar{y} = 80.90$. Take $h = 82$ and $k = 81$.

	x	y	$x-h$	$y-k$	$(x-h)^2$	$(y-k)^2$	$(x-h) \times (y-k)$
1	91	92	9	11	81	121	99
2	90	86	8	5	64	25	40
3	88	90	6	9	36	81	54
4	81	70	−1	−11	1	121	11
5	65	80	−17	−1	289	1	17
6	90	85	8	4	64	16	32
7	93	85	11	4	121	16	44
8	92	86	10	5	100	25	50
9	90	80	8	−1	64	1	−8
10	92	86	10	5	100	25	50
11	60	70	−22	−11	484	121	242
12	80	76	−2	−5	4	25	10
13	88	90	6	9	36	81	54
14	72	74	−10	−7	100	49	70
15	86	80	4	−1	16	1	−4
16	82	78	0	−3	0	9	0
17	76	80	−6	−1	36	1	6
18	75	77	−7	−4	49	16	28
19	65	80	−17	−1	289	1	17
20	89	73	7	−8	49	64	−56
	1645	1618	5	−2	1983	800	756

$\Sigma(x-h) = nd_1$ gives $5 = 20d_1$, and $d_1 = \frac{1}{4}$.
$\Sigma(y-k) = nd_2$ gives $-2 = 20d_2$, and $d_2 = -\frac{1}{10}$.
$\Sigma(x-h)^2 = n\sigma_x^2 + nd_1^2$ gives $1983 = 20\sigma_x^2 + 1\frac{1}{4}$, and $\sigma_x^2 = 99.0875$.

$\Sigma(y - k)^2 = n\sigma_y^2 + nd_2^2$ gives $800 = 20\sigma_y^2 + .20$, and $\sigma_y^2 = 39.99$.
$\Sigma(x - h)(y - k) = np + nd_1d_2$ gives $756 = 20p - .5$, and $p = 37.825$.

$$r = \frac{p}{\sigma_x\sigma_y} = \frac{37.825}{\sqrt{39.99 \times 99.8075}} = \frac{37.825}{62.95} = .601.$$

The regression equations are

$$y' = b_1x' = \frac{p}{\sigma_x^2}x' = \frac{37.825}{99.0875}x' = .381x', \text{ or } y - 80.9 = .381\,(x - 82.25).$$

$$x' = b_2y' = \frac{p}{\sigma_y^2}y' = \frac{37.825}{39.99}y' = .947y', \text{ or } x - 82.25 = .947\,(y - 80.9).$$

Exercise 123

The grades of 15 students are shown in the following table, x in history, y in mathematics, z in economics, w in foreign language.

Student	x	y	z	w
1	100	88	92	82
2	96	85	90	84
3	95	80	100	86
4	92	76	86	100
5	90	75	88	96
6	88	70	85	92
7	86	100	96	80
8	85	86	84	90
9	82	92	70	72
10	80	95	76	78
11	78	90	74	70
12	76	70	80	88
13	75	86	94	85
14	72	82	82	75
15	70	72	78	76

1. Find the values of $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{w}$.

2. Find the values of σ_x^2, σ_y^2, σ_z^2, σ_w^2.

3. Find the values of p_{xy}, p_{xz}, p_{xw}, p_{yz}, p_{yw}, p_{zw}.

4. Find the values of r_{xy}, r_{xz}, r_{xw}, r_{yz}, r_{yw}, r_{zw}.

5. Find the regression equations for (x, y), (x, z), (x, w), (y, z), (y, w), (z, w).

176. Semilog and log paper. A table of values of x and y when plotted on semilog or on log paper may give a set of points that cluster about a line. Best lines, in the sense of least square solutions, may be found for such cases by using the same methods as for ordinary ruled paper.

(a) The equation of a straight line on semilog paper, y logarithmic, is $\log y = a + bx$. Therefore, form a new column, $Y = \log y$, and find a least square solution for the table (x, Y).

(b) For a straight line on semilog paper, x logarithmic, the equation is $\log x = a + by$. Form a column, $X = \log x$, and find a least square solution for the table (X, y).

(c) For a straight line on log paper, the equation is $\log y = a + b \log y$. Use the columns, $X = \log x$, $Y = \log y$.

That is, add to the given table, (x, y), the columns $X = \log x$ and $Y = \log y$. Then on ordinary ruled paper plot the points for (x, y), (x, Y), (X, y), (X, Y). If any one of the four graphs gives a set of points that cluster about a line, a least square solution will give a satisfactory relation between x and y.

177. Best point. The method of least squares is also applicable to problems in which a number of linear equations of the form $P = Ax + By + C = 0$ are given, and it is required to find a pair of values of x and y which will best fit all the equations. Since the equations are all represented by straight lines, the problem is the same as that of finding the coördinates of a point (x, y) which best fits all the lines. In general, the required point may not be on any one of the given lines, and the substitution of its coördinates in the equation $P = Ax + By + C = 0$ will give forms such as

$$P_1 = A_1x + B_1y + C_1,$$

$$P_2 = A_2x + B_2y + C_2,$$

and so on. P_1, P_2, . . . , may be + or −, but their squares are all + and ΣP^2 can be 0 only if each P is 0, or if all the lines pass through a common point. Therefore the nearer ΣP^2 is to 0, the more nearly will the required point lie on all the lines.

The least square solution is obtained by making the conditions that $\Sigma P^2 = \Sigma(Ax + By + C)^2$ shall be a minimum. In this case, x and y are the unknown constants, and the expansion of ΣP^2 gives a quadratic in x and y.

Expanded with reference to x,

$$\Sigma P^2 = \Sigma A^2x^2 + \Sigma 2Ax\,(By + C) + \Sigma(By + C)^2,$$

which is a minimum when

$$x = -\frac{2\Sigma A\,(By + C)}{2\Sigma A^2},$$

or when

$$x\Sigma A^2 + \Sigma ABy + \Sigma AC = 0,$$

or when

$$\Sigma AP = 0,$$

where A is the coefficient of x in

$$P = Ax + By + C.$$

Similarly, if expanded with reference to y, the condition for ΣP^2 to be a minimum is $\Sigma BP = 0$.

The values of x and y are now found from the simultaneous normal equations $\Sigma AP = 0$ and $\Sigma BP = 0$, or

$$x\Sigma A^2 + y\Sigma AB + \Sigma AC = 0,$$

and

$$x\Sigma AB + y\Sigma B^2 + \Sigma BC = 0.$$

How good the solution is may be judged by calculating the value of

$$\Sigma P^2 = \Sigma CP = x\Sigma AC + y\Sigma BC + \Sigma C^2.$$

Similarly, a set of values of x, y, z which best fit n equations of the form

$$P = Ax + By + Cz + D = 0$$

is found from the normal equations,

$$\Sigma AP = \Sigma BP = \Sigma CP = 0.$$

Illustration

Find a least square solution of the equations

$$x + y - 3 = 0$$
$$2x - 3y + 4 = 0$$
$$3x - 2y + 5 = 0$$
$$3x + 2y - 6 = 0.$$

Prepare the tabulation

A	B	C	A^2	AB	AC	BC	B^2	C^2
1	1	−3	1	1	−3	−3	1	9
2	−3	4	4	−6	8	−12	9	16
3	−2	5	9	−6	15	−10	4	25
3	2	−6	9	6	−18	−12	4	36
9	−2	0	23	−5	2	−37	18	86

The given equations are of the form $P = Ax + By + C = 0$.

The normal equations are $\Sigma AP = 0$ and $\Sigma BP = 0$, or $23x - 5y + 2 = 0$ and $-5x + 18y - 37 = 0$.

The solution of these normal equations is

$$x = \frac{149}{389}, \quad y = \frac{841}{389},$$

and

$$\Sigma P^2 = \frac{149}{389} \times 2 - \frac{841}{389} \times 37 + 86 = \frac{2635}{389} = 6.8.$$

If we took $x = \frac{1}{2}$, $y = 2$, we would obtain

$$P_1 = -\tfrac{1}{2}, \quad P_2 = -1, \quad P_3 = 2.5, \quad P_4 = \tfrac{1}{2}, \quad \Sigma P^2 = 7\tfrac{3}{4}.$$

Exercise 124

Find the least square solution for each of the following sets of equations and calculate the value of ΣP^2 for each solution.

1.
$$\begin{aligned} 2x + y - 5 &= 0 \\ x - 3y - 4 &= 0 \\ 2x + 3y - 6 &= 0 \end{aligned}$$

2.
$$\begin{aligned} x + y - 2 &= 0 \\ x - y + 4 &= 0 \\ 2x + y - 3 &= 0 \\ 3x - 2y - 6 &= 0 \end{aligned}$$

3.
$$\begin{aligned} x - y + 2 &= 0 \\ x + y - 4 &= 0 \\ 2x - y - 7 &= 0 \\ -2x + 3y + 1 &= 0 \\ 3x + y - 8 &= 0 \end{aligned}$$

4.
$$\begin{aligned} x + 0y - 2 &= 0 \\ 0x - 3y + 6 &= 0 \\ 5x - y - 4 &= 0 \\ 3x + 2y - 12 &= 0 \end{aligned}$$

5.
$$\begin{aligned} 3x + y &= 7 \\ -x + y &= 3 \\ 2x - 3y &= -6 \\ 3x - y &= 0 \end{aligned}$$

6.
$$\begin{aligned} 2x - y - 7 &= 0 \\ -2x + 3y + 1 &= 0 \\ 3x + y - 8 &= 0 \\ 0x + y + 6 &= 0 \\ 3x + 0y - 10 &= 0 \end{aligned}$$

7.
$$\begin{aligned} 3x + 4y - 2z &= 5 \\ x - y - z &= 4 \\ 2x - 2y + 3z &= 6 \\ 5x + y - 5z &= 2 \\ 4x - 3y + 2z &= 3 \end{aligned}$$

8. Show that, if in example 1 the three equations are multiplied throughout, the least square solution will be: (a) unchanged if the three multipliers are the same; (b) different if the three multipliers are not the same.

9. In examples 1 - 6, draw the lines represented by the equations and indicate the point that represents the least square solution.

178. Practical application. A company manufactures two articles, A and B, each of which goes through two processes, C and D, the costs in processes C and D being different for the two articles. The table in example 1, Exercise 125, shows the number of units of A and B turned out weekly and the weekly payrolls in shops C and D. It is required to find the cost of labor to be allocated to each article for each process.

Let x = number of dollars per unit A in shop C.

Let y = number of dollars per unit B in shop C.

Then for the first week, $10x + 4y = 40$, and similarly for other weeks. Ten equations of the form $ax + by = C$ are

made, and a least square solution of x and y is required. Similarly, 10 equations are made for shop D.

Exercise 125

1.

Week	*Articles* A	*Articles* B	*Payroll* C	*Payroll* D
1	10	4	40	40
2	12	5	45	50
3	8	4	38	43
4	9	6	40	50
5	11	5	43	40
6	15	8	60	72
7	7	9	40	60
8	12	7	50	60
9	10	6	41	55
10	14	8	59	70

(a) Show that in shop C a unit of A costs \$3.02 and a unit of B costs \$2.05.

(b) Show that the unit costs in shop D are \$1.91 and \$5.36.

2. A restaurant buys butter and eggs from Roberts & Co. in varying quantities and at varying prices from month to month. A single bill is rendered each month. Thus in January 1000 lbs. butter at 40¢ a pound and 500 dozen eggs at 30¢ a dozen were used. The total bill for January was \$400 + \$150 = \$550.

Complete the table for 12 months, taking arbitrary figures, using the form

Month	*A (butter)*	*B (eggs)*	*C (total cost)*
1	1000	500	550

(a) From the table, find a least square solution giving the probable average cost per pound of butter, x, and per dozen eggs, y.

(b) From the original data that were assumed for butter, find the mean cost per pound.

(c) Similarly, find the mean cost per dozen eggs.

Note: The values found in (b) and (c) may differ considerably from those found in (a). The reason for the differences is that many values of

the prices for a pound of butter and for a dozen eggs for the month of January are possible so as to give a total cost of \$550. The same statement is true for each subsequent month. Hence the results of (b) and (c) are subject to very wide variations. The least square solutions are probably better estimates of the average cost per pound of butter and per dozen eggs than are the results of (b) and (c).

CHAPTER XV

APPROXIMATE CALCULATIONS

179. Approximate numbers. When a measurement is made, the number that represents the measurement rarely corresponds exactly to the magnitude of the thing measured. How nearly it does correspond depends upon the precision of the instruments used and upon the precision required in the particular case. If the distance between two railroad stations is stated as 12.7 miles, we mean that the measurement was made to the nearest tenth of a mile and that the true distance in miles may be anywhere between 12.65 and 12.75, a result usually written $12.7 \pm .05$. Similarly, if the distance is stated as 12.70, the measurement was made to the nearest hundredth of a mile, and the true distance is $12.70 \pm .005$.

Scientists write all approximate numbers in the form $10^n c$, where n is a positive or a negative integer and c is a number between 1 and 10. We then say that the number is written to as many significant figures as there are figures in c. Thus

$$68300 = 6.8300 \times 10^4,\ 683.0 = 6.830 \times 10^2,\ 0.0604 = 6.04 \times 10^{-2},$$

are numbers of 5, 4, and 3 significant figures, respectively. An exact number is correct to as many significant figures as we please.

The exponent of 10 in $N = 10^n c$ is called the *order of the number* N. Thus the number $603.2 = 6.032 \times 10^2$ is of the second order.

Approximate numbers may arise, not from measurement, but from calculations. Most tables such as square root,

logarithms, compound interest, and so on consist of numbers that are correct only to a certain number of decimal places, the last figure being the nearest figure for that decimal place.

When calculations are made from numbers obtained by measurement or from tables, we must remember that the numbers are approximate, and we should not make a pretense at accuracy that is not justified.

Thus 12.7 miles, when expressed in feet, appears to give $12.7 \times 5280 = 67{,}056$. But if 12.7 means $12.7 \pm .05$, the number of feet is $5280\ (12.7 \pm .05) = 67{,}056 \pm 264$. That is, the distance in feet is between 66,792 and 67,320 and is approximately 67,000.

180. Addition. When two numbers of different orders are to be added, one of them being given to the nearest unit and the other to the nearest hundredth, it is evident that the sum is correct at most to the nearest unit. In general, a sum cannot be more accurate than the least accurate of the addends.

181. Subtraction. As in addition, the result of subtraction can be as accurate at most as the less accurate item. If two numbers of the same order are each correct to 6 significant figures, their sum may be correct to as many as 6 significant figures, but their difference may be correct only to one or two figures. Thus $3.27458 - 3.27369 = .00089$.

182. Multiplication. If M has 6 significant figures and the decimal point is disregarded, M is written 10^5a, the true value being $10^5a \pm \frac{1}{2}$. Similarly, if N has 6 significant figures, it is written 10^5b, the true value being $10^5b \pm \frac{1}{2}$. The product $M \times N$ is not greater than

$$(10^5a + \tfrac{1}{2})\ (10^5b + \tfrac{1}{2}) = 10^{10}ab + \tfrac{1}{2}(a + b)10^5 + \tfrac{1}{4},$$

or less than

$$(10^5a - \tfrac{1}{2})\ (10^5b - \tfrac{1}{2}) = 10^{10}ab - \tfrac{1}{2}(a + b)10^5 + \tfrac{1}{4}.$$

The term $\frac{1}{4}$ may be disregarded and we conclude that the product $M \times N$ is not the same as $(10^5a)(10^5b)$ but may vary from it in either direction by as much as $\frac{1}{2}(a + b)\ 10^5$.

Since a and b are each between 1 and 10, $\frac{1}{2}(a + b) < 10$, and $\frac{1}{2}(a + b)10^5$ is a number of 6 figures.

If $ab < 10$, $(10^5a)(10^5b) = 10^{10}ab$ is a number of 11 figures; if $ab > 10$, $10^{10}ab$ is a number of 12 figures.

In either case, $10^{10}ab \pm \frac{1}{2}(a + b)10^5$ shows that the last 6 figures of $10^{10}ab$ may be unreliable and that the product is correct at most to 6 significant figures—that is, to 5 or to 6 figures.

The student may show that, if one of two factors is correct to 5 figures and the other to 7, the product is correct to 5 figures at the most.

183. Division. Suppose P and Q are 6-figure numbers which, with the decimal point disregarded, are written 10^5a and 10^5b. The quotient $10^5a/10^5b$ or a/b may be written to as many figures as we please, say 10, the decimal point appearing before or after the first figure, depending upon whether a is less or greater than b. Let us see how many of these figures are reliable.

The true values of P and Q are $10^5a \pm \frac{1}{2}$ and $10^5b \pm \frac{1}{2}$. The true quotient

$$\frac{P}{Q} = \frac{10^5a \pm \frac{1}{2}}{10^5b \pm \frac{1}{2}}$$

is not less than

$$\frac{10^5a - \frac{1}{2}}{10^5b + \frac{1}{2}},$$

and is not greater than

$$\frac{10^5a + \frac{1}{2}}{10^5b - \frac{1}{2}}.$$

The calculated quotient $\dfrac{a}{b}$ differs from these limits by

$$\frac{a}{b} - \frac{10^5a - \frac{1}{2}}{10^5b + \frac{1}{2}} = \frac{\frac{1}{2}(a+b)}{b(10^5b + \frac{1}{2})} = \frac{\frac{1}{2}\left(\frac{a}{b} + 1\right)}{10^5b + \frac{1}{2}}$$

and

$$\frac{10^5a + \frac{1}{2}}{10^5b - \frac{1}{2}} - \frac{a}{b} = \frac{\frac{1}{2}\left(\frac{a}{b} + 1\right)}{10^5b - \frac{1}{2}}.$$

If $a < b$,

$$\frac{1}{2}\left(\frac{a}{b} + 1\right) < 1.$$

If $a > b$,

$$\frac{1}{2}\left(\frac{a}{b} + 1\right) < 6.$$

In either case, the first figure in the difference will not be affected, whether we divide by 10^5b or by $10^5b \pm \frac{1}{2}$. Hence the first figure in the difference between the calculated quotient and the true quotient is

$$\frac{\frac{1}{2}\left(\frac{a}{b} + 1\right)}{10^5b} = \frac{\frac{1}{2}\left(\frac{a}{b} + 1\right)}{b} \times 10^{-5}.$$

If $a < b$, the first figure in the difference may be a figure in the 6th decimal place, and, if $a > b$, it may be a figure in the 5th decimal place.

We conclude, therefore, that only the first 5 figures of the quotient $\dfrac{10^5a}{10^5b}$ are reliable.

In general, if two numbers are each given correctly to n figures, neither their product nor their quotient can be correct to more than n figures. The results of abbreviated multiplication and division, pages 120 and 123, are more accurate than the results of ordinary multiplication and division whenever approximate numbers are involved.

184. Square root. If N is an exact whole number of 7 or 8 figures, the rules for square root enable us to find 10^3a,

the first 4 figures of $\sqrt{N}$, and $\sqrt{N} = 10^3a + b$ where $b < 1$. Then

$$N = 10^6a^2 + 2ab10^3 + b^2,$$

and

$$\frac{N - 10^6a^2}{2a10^3} = b + \frac{b^2}{2a10^3}.$$

$N - 10^6a^2$ is the remainder after the first 4 figures of N have been found and $\frac{N - 10^6a^2}{2a10^3}$ is the result of dividing this remainder by the usual trial divisor, twice the root already found. Ordinary division gives a result greater than b by $\frac{b^2}{2a10^3}$, which is less than .0005. That is, after the square root has been found to 4 figures, the following 3 figures may be found by dividing the remainder by twice the root.

Now if N is not an exact number but is given accurately to 7 or 8 figures only, its true value is $N \pm \frac{1}{2}$ and

$$\frac{N - 10^6a^2}{2a10^3} = b + \frac{b^2}{2a10^3} \pm \frac{.5}{2a10^3}.$$

Since $\frac{b^2 \pm .5}{2a10^3}$ is less than $\frac{1 \pm .5}{2 \times 10^3}$ or less than .0008, we conclude that the first 7 figures of $\sqrt{N}$ are reliable, but that the 8th figure may not be reliable. Similar conclusions may be drawn from an analysis of cube root, and the following general statement may be made:

If numbers in tables are accurate to n figures, the results of the operations, multiplication, division, root extraction, and so forth, may not be accurate to more than n figures and only the first $n - 1$ figures are reliable.

Since raising a number to a power requires a number of multiplications, the accuracy of the result decreases as the power increases. Thus if the value $x = 2^{100}$ is found by logarithms, $\log x = 100 \log 2 = 30.10$, we can be sure

of only the first two figures, 12, when 4-place tables are used, and the best estimate is that $2^{100} = 1.2 \times 10^{30}$.

185. Solution of equations. The principles thus far established are applicable to the solutions of equations in which the coefficients are approximate numbers. The degree of accuracy of the final solution depends upon the number of operations that are necessary. Certainly the solution is not accurate to a greater number of figures than that of the least accurate of the coefficients.

186. Graphic calculations. Many results of calculation may be found approximately from a straight line graph.

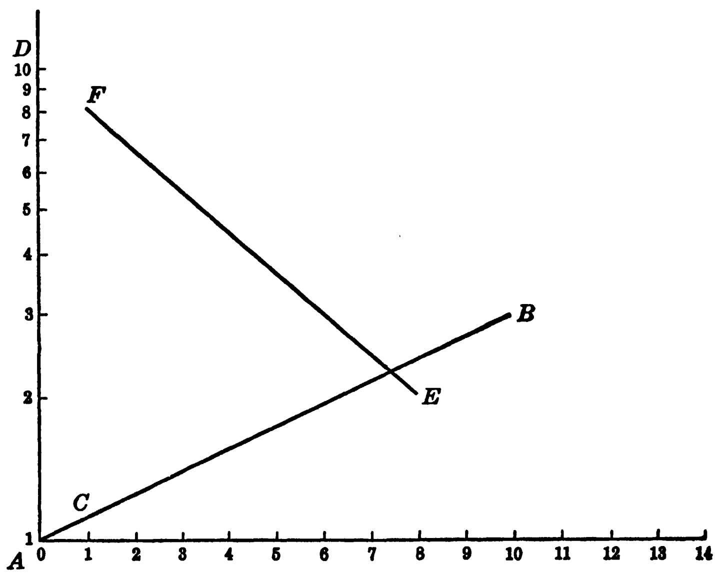


Fig. 45.

The line itself need not be drawn. Merely place a ruler so as to join two points that are located and read off the required results. The larger the units and their subdivisions, the better will be the approximation.

The graphs of $y = (1 \pm r)^x$ are straight lines on semilog paper (S L) and pass through the origin $x = 0$, $y = 1$ so that only one additional point is to be located.

Illustrations

1. At what rate of interest will \$1 become \$3 at the end of 10 years?

In the equation $y = (1 + r)^x$, $x = 10$ and $y = 3$.

In Fig. 45, page 309, the scale on y shows 1 at A and 10 at D. Locate point B at $x = 10$, $y = 3$, and join B with A. On the line AB, $x = 1$ shows $y = 1.11$. Hence $r = .11$, and the rate is approximately 11% per annum.

2. At the end of 8 years, the book value of a machine is 20% of its initial value. If the depreciation was written off by the reducing balance method, what was the annual rate?

In the equation $y = (1 - r)^x$, $x = 8$ and $y = .2$.

In Fig. 45, the scale on y is .1 at A and 1 at D. Locate E at $x = 8$, $y = .2$, and join D with E. On the line DE, $x = 1$ shows $y = 1F = 1 - r = .82$. Hence $r = .18$, and the annual rate of depreciation is approximately 18%.

3. The book value of the machine of example 2 at the end of any number of years is easily found. Thus at the end of 5 years the book value is approximately 37%.

187. The slide rule. When it is necessary to perform a large number of multiplications and divisions, logarithms are ordinarily employed. Four-place tables enable us to obtain accuracy to four figures at the most. The slide rule generally used by engineers enables us to perform such calculations much more rapidly, but the accuracy is dependable to three figures at the most.

The slide rule consists of two rulers on which scales are marked, and one ruler slides along the other.

Fig. 46 represents two rulers, A and B, each marked with the ordinary or standard scale, ruler B being placed so that its zero mark is under the mark 3 on A. Any number on B, say 5, corresponds to $3 + 5$, or 8, on A. Thus it

is possible, with two ordinary rulers, to perform additions and subtractions mechanically. To find $3 + 5$, place

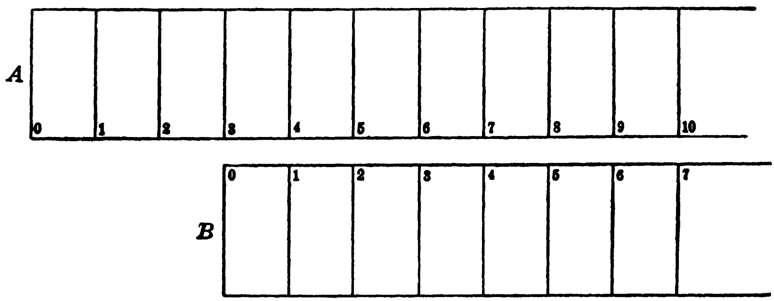


Fig. 46.

zero of B under 3 of A. The number on A above 5 of B is the sum, 8. To find $8 - 5$, place 5 of B under 8 of A. The number on A above zero of B is the difference, $8 - 5$, or 3.

Fig. 47 represents two rulers, A and B, each marked with the logarithmic scale. A number, 5, on either scale

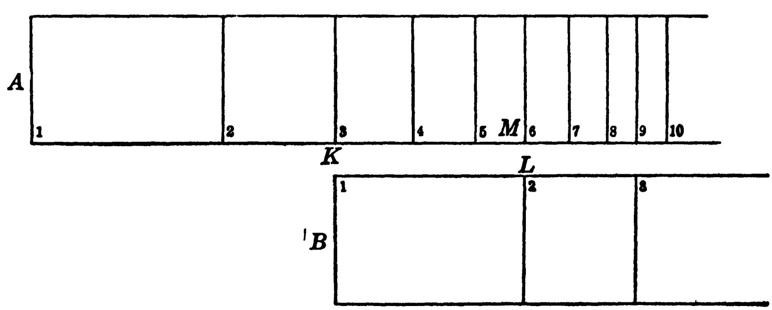


Fig. 47.

is at a distance from the starting point equal to log 5, or 0.6990 of the unit which extends from mark 1 to mark 10. Mark 1 is at a distance of log 1, or zero, from the starting point; that is, mark 1 is the starting point.

The setting shown in Fig. 47 gives, as in the preceding illustration, $AK + BL = AM$, which in this case is really $\log 3 + \log 2 = \log 6$, or $\log (3 \times 2) = \log 6$. But since the word *log* is not written on the scale, the setting is read "$3 \times 2 = 6$", which is found as follows: Place the starting point marked 1 of B under 3 of A. The number on A above 2 of B is 3×2, or 6.

To find $6 \div 3$, place 3 of B under 6 of A. The number on A above mark 1 of B is the quotient, 2.

The pocket slide rule is usually 10 inches long and contains two units. Since each unit is 5 inches long, the scale divisions are not unduly crowded. Multiplications and divisions can be performed with such a slide rule as accurately as if three-place logarithm tables were used. The value of a series of multiplications and divisions can be obtained by means of a slide rule in a few seconds.

188. Stirling approximation. When n is a large number, more than 10, an approximate value of $n!$, called the Stirling approximation is given by

$$n! = n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}.$$

By using this approximation, the expression on page 233 becomes

$$\frac{400^{400+\frac{1}{2}} e^{-400} \sqrt{2\pi}}{100^{100+\frac{1}{2}} e^{-100} \sqrt{2\pi}\, 300^{300+\frac{1}{2}} e^{-300} \sqrt{2\pi}} \times \frac{3^{300}}{4^{400}}$$

which reduces to $\dfrac{1}{\sqrt{150\pi}} = .046.$

ANSWERS TO EXERCISES

ANSWERS TO EXERCISES

EXERCISE	PAGE	
2.	5	(1) 809, 363, 66; (2) $10t + u$, $t + u$, tu; (3[g]) $(x + y)(x - y) = x^2 - y^2$; (3[j]) $(x + y)^2 - (x - y)^2 = 4xy$.
3.	9	(1) $4\frac{1}{6}$; (2) $\frac{102}{35}$; (3) $\frac{4}{9}$; (4) $\frac{29}{90}$; (5) 3; (6) $\frac{11}{15}$; (7) $\frac{150}{119}$; (8) $4\frac{2}{3}$; (9) 8; (10) $3\frac{3}{4}$; (11) $2\frac{1}{3}$; (12) 1; (13) $1\frac{3}{4}$; (14) $\frac{11}{8}$; (15) $\frac{7}{52}$; (16) $\frac{23}{12}$.
4.	10	(5) 3000, 160, 1800, 84, 120; (6) $12\frac{1}{2}\%$, $11\frac{1}{9}\%$; (7) $\$11\frac{1}{9}$; (8) 20%; (9) $\$15$; (10) 5%, $4\frac{16}{21}\%$; (11) 5%, $4\frac{16}{21}\%$; (12) $11\frac{1}{9}\%$; (13) $\$210$; (14) decrease 1%; (15) 500.
5.	12	(1) $\$157.50$; (2) $68\frac{1}{2}\%$; (3) less by $\$67.50$; (4) more by $\$12.60$; (5) unchanged; (6) 37%, 55%, 65%; (7) $28\frac{4}{7}\%$; (8) $20\frac{40}{63}\%$.
6.	13	(1) Aug. 20, Sept. 20, Oct. 20, Aug. 19, Sept. 18, Oct. 18; (2) 53, 54; 108, 109; 90, 92; (3) 1800, 1898, 1900, 1914.
7.	13	(1) $\$1176$, 1200, 816.33, 383.67, March 31; (2) $\$3913.40$, Apr. 15; (3) $\$387.12$.
8.	15	(1) $\$16$, $13\frac{1}{3}$, 12, $10\frac{2}{3}$, $9\frac{1}{3}$; (2) $\$727.08$; (3) $\$322.08$; (4) 6%; (5) $\$924$.
9.	17	(1) $I = \$150$, 156.25, 127.63, 54.38, 71.50, $E = \$147.95$, 154.11, 125.88, 53.64, 70.52; (2) $\$365$, 360; (3) $\$270$, 5.4%; (4) $\$360$, $\frac{5}{18}$; (5) $\$1080$, 18,000; (6) $\$41.25$, 59.18, 5%, $6\frac{1}{12}\%$, $\$7285.71$, 8984.62, 68 days, 146 days; (7) gains $\$97.10$, 8.98%.
10.	19	(1) Proceeds: $\$5016.94$, 2722.50, 7511.88, 8028.38, 14,750; (2) $\$11.42$; (3) $\$6505.91$, May 20, 5%; (4) 5.08%, 6.06%, 4.04%; (5) 5%.

EXERCISE	PAGE	
11.	21	(1) True discount: \$9.90, 16.39, 19.61, 25.97, 32.89, 22.03, 16.56, 11.06; (2) \$5264, 7.896%, 79 days.
12.	22	(1) June 4; (2) \$2008.67, 1995; (5) \$538.46; (6) $21\frac{1}{19}$%; (7) 20%; (8) April 13, \$996.83, 1006.17.
13.	26	(1) 30, 12, −4, −18, −30, 1, 0, −5.02, −2, −1; (2) 1.126, −1.126, −2.36428, .31563, −.31563, $\frac{1}{12}$, $-\frac{1}{12}$.
14.	27	(1) 6, +2, 12, 22, −12, −22, 0, −20, −3.4, $-\frac{1}{2}$; (2) −2, −2, −4.425, +4.425, $-2\frac{7}{8}$, +1, −1, $+\frac{1}{2}$.
15.	29	(1) $-x$; (2) $-2a-7b$; (3) $-7a+ax+7x$; (4) x^3-2x^2+2x; (5) $4ab-2ac+9bc$; (6) $\frac{1}{4}a$; (7) $-7.3x$; (8) $1.3x^2$; (9) $-398a$; (10) $-809x^2$; (11) $3a-3b$, $-a-3b$, $2x-3y-2a+3b$, $+6b$, $-4a$, $-4a+6b$; (12) $4a-4b$, $6a-6b+6c$, $-2b+8c$, $2a-2c$, $-6a+6b-6c$, $2b-8c$.
16.	30	(1) $12-x$, 1, $4x-3y$, $2x+1$, 9.
17.	31	(1) $12a^4$; (2) $60a^2bc$; (3) a^3b^2cx; (4) $8a^7b^3c^2$; (5) $\frac{1}{3}a^3x^3$; (6) $3a^5x^3$.
18.	33	(1) $6x^2-17x+12$; (2) $6x^3-13x^2-6x+8$; (3) $2x^4-7x^3+8x^2-x-12$; (4) x^3-1; (5) x^3+1; (6) x^4+x^2+1; (7) $x^4-10x^3+35x^2-50x+24$; (8) $-12x^3+72x$.
19.	35	(1) $5a^3$; (2) $6a^3$; (3) $-6x^2$; (4) $-6x^3+5x^2+4x$; (5) $-a+6b^2$; (6) $x-1$; (7) $2a^2-4ab^2$; (8) $3x^2y^2-xy-2xy^2$; (9) x^3-1; (10) $a+b-c$.
20.	37	(1) $3x+2$; (2) $3x-2$; (3) $9x^2+6x+4$; (4) quotient $9x^2-6x+4$, remainder -16; (5) $9x^2-6x+4$; (6) quotient $9x^2+6x+4$, remainder $+16$; (7) $2x^2+3x+2$; (8) x^2-1; (9) x^2+3x+2; (10) x^2-2x+1.

EXERCISE	PAGE	
22.	41	(1) $5 \cdot 3^3 \cdot 11^2$; (2) $5 \cdot 3^3 \cdot 11 \cdot 13$; (3) $3^3 \cdot 7^2 \cdot 11$; (4) $3^2 \cdot 11^3 \cdot 13$; (5) $3^5 \cdot 5 \cdot 7 \cdot 11^2$; (6) $2^2 \cdot 3 \cdot 11^2 \cdot 19$; (7) $3 \cdot 5^2 \cdot 11^2 \cdot 19$; (8) $5 \cdot 11 \cdot 13 \cdot 181$; (9) $5 \cdot 3^3 \cdot 7 \cdot 11^2$; (10) $3^6 \cdot 7^2 \cdot 11^3$.
23.	42	(1) $\frac{25}{21}, \frac{11}{15}, \frac{3}{5}, \frac{4}{99}, \frac{1}{3}, \frac{11}{12}$; (2) $\frac{29}{216}, \frac{422}{1617}, \frac{1}{156}, \frac{8}{63}, \frac{5}{6}, \frac{8}{495}$.
24.	43	(2) $5x^2(3x+5)(3x-5)$, $16x^3(3x-5)^2$, $7(x+2)(x-2)$, $3(x-2)^2$, $9a(x^2+1)(x+1)(x-1)$, $5ax^3(x+3)^2$; (3) $\frac{2x}{a}, \frac{x^2}{4a}, \frac{18}{5}, \frac{x+3}{x-3}, \frac{5x+1}{x^3-x}, \frac{3(a+x)}{5x(a-x)}$.
25	46	(1) $(x-3)(x+3)$, $(x-2)^2$, $(x+3)^2$; (2) $(x-2)(x-1)$, $(x+2)(x+1)$, $(x-2)(x+1)$, $(x+2)(x-1)$; (3) $(x+1)(x^2-x+1)$, $(x-2)(x^2+2x+4)$, $(x+3)(x^2-3x+9)$, $(x-4)(x^2+4x+16)$; (4) $(x-2)(x-3)(x-4)$; (5) $(x-2)(x+3)(x^2+x+1)$; (6) $\frac{x-3}{x+2}$; (7) $\frac{x^2-4}{x^2-1}$; (8) $\frac{2x}{(x+2)(x^3-8)}$; (9) $\frac{2}{(x+2)(x+4)}$; (10) $\frac{x+3}{x+6}$.
26.	49	(1) $x = 7$; (2) $x = 4$; (3) $x = 1\frac{1}{2}$; (4) $x = \frac{7}{4}$; (5) $x = 3$; (6) $x = 2$; (7) $a = \frac{14}{5}$; (8) $a = 2$; (9) $x = 2$; (10) $a = 5$; (11) $x = \frac{5}{2}$; (12) $x = \frac{4}{3}$; (13) $x = \frac{1}{4}$; (14) $x = \frac{12}{5}$; (15) $x = \frac{9}{2}$; (16) $x = -1$; (17) $x = -\frac{10}{9}$; (18) $x = \frac{7}{6}$; (19) $x = -1$; (20) $x = +1, -1, +2$; (21) $x = -2, -3, -4$; (22) $x = \frac{a-5}{a-3}$, $a = \frac{3x-5}{x-1}$; (23) $a = \frac{5h-2}{h+3}$, $h = \frac{3a+2}{5-a}$; (24) $x = \frac{3+3y}{2-y}$,

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26.	49	$y = \dfrac{2x-3}{x+3}$; (25) $x = \dfrac{b(y+1)}{5(y-1)}$, $y = \dfrac{b+5x}{5x-b}$, $b = \dfrac{5x(y-1)}{y+1}$; (26) $p = \dfrac{s}{1+rt}$, $r = \dfrac{s-p}{pt}$, $t = \dfrac{s-p}{pr}$; (27) $m = \dfrac{p}{1-rt}$, $r = \dfrac{m-p}{mt}$, $t = \dfrac{m-p}{mr}$.
27.	50	(1) 23, 77; (2) 23¢, 77¢; (3) 32¢, 64¢; (4) 46; (5) 59; (6) $\frac{13}{17}$; (7) 6 quarters; (8) 25 of 15¢ kind; (9) \$6000 at 4%; (10) 11, 22; (12) 5, 6, 7 or -6, -5, -4; (13) $x = \dfrac{a}{(1-a)}$, $a = \dfrac{x}{(1+x)}$; (14) $s = p + prn$; (18) $\dfrac{n(c-b)}{(a-b)}$, $\dfrac{n(a-c)}{(a-b)}$; (19) 2 or 3; (20) $\frac{1}{6}$, $\frac{1}{9}$.
28.	54	(1) 5, 2; (2) $\frac{3}{2}$, $-\frac{1}{4}$; (3) 10, 3; (4) 5, 1; (5) 1, 1; (6) 5, $\frac{1}{2}$; (7) 5, 3; (8) 3, 1; (9) -9, 7; (10) -8, -1; (11) 8, 2, 1; (12) 4, 3, 2; (13) 5, 2, 3; (14) 6, 2, 3; (15) 4, 8, 9; (16) $2\frac{1}{2}$, $1\frac{1}{2}$, $\frac{1}{2}$; (17) 3, -2; (18) 2, -3, -1; (19) 1, 0, -3, 5; (20) 3, 2, 4, 1.
29.	55	(1) inconsistent; (2) dependent; (3) dependent; (4) dependent; (5) inconsistent.
30.	57	(1) 75; (2) 35, 30, 5; (3) \$1.71, 2.79, .50; (4) 437; (5) 49; (6) 83; (7) 8, 10, 14.
31.	62	(1) $2x^2 - 3x + 4$, $x^2 - x + 1$, $x^3 - 4x^2 - 2x + 3$.
32.	64	(1) 3, $-\frac{1}{2}$; (2) 0.257, -2.591; (3) 0.558, $-.358$; (4) 1.618, $-.618$; (5) 2.186, $-.686$; (6) 1.000, 1.500; (7) .618, -1.618; (8) 3.791, $-.791$; (9) .865, $-.643$; (10) 5.702, $-.702$; (11) 1, 2.303, -1.303; (12) 2, 3, 2.562, -1.562.
33.	66	(1) 4, 4, -1, 1, $\frac{5}{3}$, $-\frac{2}{3}$, 1, 1, $-\frac{5}{3}$, $-\frac{2}{3}$, $\frac{7}{5}$, $\frac{4}{5}$; (2) $x^2 - 5x + 6 = 0$, $x^2 - 3x - 4 = 0$, $6x^2 -$

EXERCISE	PAGE	
33.	66	$5x + 1 = 0$, $3x^2 - 10x + 3 = 0$, $x^2 + 3x + 2 = 0$, $x^2 - 9 = 0$; (3) equal, real, real, depends on value of $4k^2 - 12k$, imaginary, imaginary, imaginary, depends on value of $9 + 4c$; (4) any value except between 0 and 3, -1; (5) not less than $-2\frac{1}{4}$, $+4$, -2; (6) not more than $5\frac{1}{3}$, 4, 4 or -16; (7) $\frac{3}{2}$ or $-\frac{2}{3}$; (8) $\frac{3}{2}$ or $\frac{2}{3}$; (9) 20% and 40%; (10) 50% and 30%; (11) 13; (12) $4 + 4\sqrt{2}$ mi. per hr.; (13) $k = 4$ roots are $\frac{3}{2}$ and $\frac{5}{2}$, $k = -64$ roots are $\frac{3}{8}$ and $-\frac{5}{8}$.
36.	77	(1) 30°, 60°, 90°; (2) $22\frac{1}{2}°$, $67\frac{1}{2}°$; (3) 105°; (4) $\frac{10}{3}$, $\frac{24}{5}$, $\frac{3}{2}$, $\frac{12}{5}$; (5) $Be = \frac{3}{5}CD$, $AE = \frac{3}{5}AD$; (6) $\frac{16}{3}$, 4; (8) 3, $\frac{24}{5}$, $7\frac{4}{5}$; $7\frac{1}{2}$ to left of D; 8 to left of C.
37.	80	(1) $9\sqrt{3}$; (2) $2\sqrt{60}$; (3) 24, $\sqrt{300}$, $\sqrt{1080}$.
38.	83	(3) 7, $\frac{14}{5}$, $(\frac{11}{2}, \frac{19}{2})$; (4[a]) (3, 6), (6, 1), (8, 4); (4[b]) $\frac{3}{2}$, $-\frac{2}{5}$, $-\frac{5}{3}$.
39.	86	(4) $2y - 3x = 4$, $y - 2x = 1$, $4y - 5x = -3$, $4y + 3x = 0$; (6) 11, -2; (7) $\sqrt{52}$.
40.	89	(5) $\frac{39}{7}$, $\frac{167}{6}$; $\frac{49}{4}$, $\frac{43}{5}$; $2\frac{10}{89}$, $55\frac{1}{2}$; $\frac{2}{5}$, $\frac{2}{11}$; 2.721, 7.417; (7) $(5, 9\frac{1}{2})$, $(4\frac{1}{3}, 8)$, $(5\frac{2}{3}, 11)$, $(4\frac{1}{2}, 8\frac{3}{8})$; (8[a]) (4, 8), (7, 6), (6, 3); (8[c]) $(5\frac{2}{3}, 5\frac{2}{3})$.
41.	91	(1) 24.9598, 7.89298, 6.97128, 22.0451; (3) 8.17822, 3.79599, 1.76194.
43.	93	(1) $-2\frac{19}{23}$, $\frac{5}{13}$; (3) $1\frac{7}{9}$; (9) 2.135, $-.468$.
44.	95	(1) $(3, -10)$, $(3,10)$, $(\frac{4}{3}, -\frac{19}{3})$, $(\frac{4}{3}, \frac{19}{3})$, $(\frac{5}{4}, -\frac{49}{8})$, $(\frac{5}{4}, \frac{49}{8})$; (3) (2, 0), the x axis is tangent to the curve; (4) (2, 11); $x = -.345$ or $+4.345$, $y = 3$; $y = 4x + 3$.
45.	97	(2) 3; (3) 27, $\frac{1}{8}$, 4, $\frac{81}{16}$, 1, $\frac{9}{4}$; (6) 5.916, .845, 2.236.
48.	104	(1) 3.2079; (3) 16.65; (5) 5.2817; (7) 0.506; (9) 197,600; (11) 0.8195; (15) 6.7538; (17) 0.006714; (18) 0.2985; (19) 97.275; (20) 0.7943; (21) 0.1436, 1.585, 0.3655, 0.7481.

EXERCISE	PAGE	
49.	105	(1) 612.0; (2) 5.996; (3) 0.0002693; (4) 562.9; (5) 0.003653; (6) 180.0; (7) 0.1807; (8) −.3335; (9) −.4886; (10) 0.2396.
50.	106	(1) 24.73; (2) .04044; (3) 2.013; (4) 0.5000; (5) 0.631; (6) 0.2460; (7) 0.1845; (8) 0.2483.
51.	107	(1) 5.64; (2) 0.4121; (3) 1.361; (4) 0.0906; (5) 0.1086; (6) .01977.
52.	108	(1) 3.162; (2) 2.154; (3) 2.449; (4) 2.450; (5) 0.9284; (6) 1.459; (7) 0.5486; (8) 2.080; (9) 4.941; (10) 371.71.
53.	109	(1) 13.42; (2) 1.5; (3) 3.318; (4) 3.162; (5) 0.82; (6) −.75; (7) $\frac{2}{3}$; (8) $\frac{1}{8}$; (9) −3; (10) 11.16.
54.	110	(1) 0.08012; (2) 34.95; (3) 0.02858; (4) 2.655; (5) 0.358; (6) 11.9; (7) 0.057; (8) .8688; (9) .9106; (10) 238.9; (11) .5871; (12) 2.291; (13) 1.222; (14) .3895.
55.	111	(1) $x = 2$; (2) $x = 2 \cdot 10^9$; (3) $x^2y^3 = 10$; (4) $x^3y^2 = 3^6$; (5) $n = 3$; (6) $x = 5$; (7) $x^2 = 10y$; (8) $y = 1000^x \cdot 10$.
56.	112	(1) 1.049; (2) 2.5441; (3) .6232; (4) 1.8114; (5) .5351; (6) 2.771; (7) 3.09; (8) 2.836; (9) 23.9; (10) .767; (11) 2.1205; (12) .4973; (13) 1.293; (14) −4.22; (15) −.237; (16) 9, 4; 16, 1; 3, 1; 13, 7.
61.	125	(1) 1.24953; (2) 4.13%; (3) 1.63305; (4) 2.14%; (5) 457.90; (6) 14.20; (7) .72650; (8) 3.93%; (9) 46.55; (10) 261.11.
62.	129	(1) 1386.51; (2) 4390.27; (3) 1635.14; (4) 2687.04; (5) 3481; (6) 1.225%; (7) 3.77%; (8) 1.79%; (9) .684%; (10) 16.48.
63.	131	(1) \$716.36; (2) Dolan pays \$15.23; (3) \$1537.77; (4) Green pays 8 cents, Brown pays \$17.67, Brown pays \$33.14; (5) \$1747.80, 1791.90; (6) \$2558.48, 1880.18; (7) \$108.87, 513.70.

EXERCISE	PAGE	
65.	136	(1) \$11,009.50; (2) \$12,405.80; (3) \$2975.50; (4) \$2566.68; (5) \$2200.11, 2382.40.
66.	137	(1) 819.54, 1644.20, 91.33, 123.06, 1009.90, 985.33, 1071.40, 10,868; (2) 2.23%, 2.28%, 2.37%, 2.28%, 1.508%, 2.591%, 1.977%, 5.063%; (3) 1172.31; (4) 2201.90, 7090.54, 11,954.66; (5) 682.54; (6) 122.44.
67.	141	(1) 2304.98, 2309.88, 2312.37, 2314.10; (2) 1722.38, 1718.78, 1716.86, 1715.62; (3) 1935.31, 1934.48, 1934.03, 1933.75; (4) 1648.73, 1644.24, 1641.87, 1640.31; (5) 2589.87, 2599.79, 2604.87, 2608.36.
68.	142	(1) 6%, 6.09%, 6.136%, 6.168%; (2) 6%, 5.9126%, 5.86956%, 5.84112%; (3) 1.46739%, 1.48892%, 1.508%; (4) .99505%, .98534%.
69.	142	(1) 224.77; (2) 8689; (3) 5209.74; (4) 191.95; (5) 1.17%; (6) 2.91%; (7) 2.75%; (8) 4724, 4063.20; (9) 3.33% per month or 48% effective annual; (10) 733.44; (11) 346.07, 350.10, 871.73, 900.74; (12) 7523.40; (13) 996.80; (14) 3.62% per month; (15) 290.30, 290.91, 291.23, 291.11, 291.44; (16) 1862; (17) 631.63; (18) *A* pays 140.73; (19) 2949.19; (20) 1189.08; (21) 7468.97; (22) 292.20; (23) Original debt required 33 payments of \$500 and a balance of \$7.16, should pay \$8886.90; (24) 18 payments, balance \$3735; (26) 131.40. 76.10, 2284, 1738, 90.63, 1616.30, 2464, 1294.
70.	150	(1) 915.32; (2) 1094.57; (3) 1061.51; (4) 952.44; (5) 996.39; (6) 1082.94; (7) 909.29; (8) 1032.64; (9) 1042.29; (10) 866.57.
71.	151	(1) 863.74; (2) 922.30; (3) 1005.10; (4) 1066.10; (5) 1038.60; (6) 1041.20; (7) 1059.70; (8) 1104.70; (9) 1164.60; (10) 994.90.

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72.	153	(1) 2.5%; (2) 3.19%; (3) 1.90%; (4) 1.91%; (5) 2.60%; (6) 1.71%; (7) 2.20%; (8) 2.85%; (9) 1.85%; (10) 2.10%; (11) 2.54%.
73.	154	(1) 1091.69; (2) 1064.44; (3) 830.13; (4) 887.38; (5) 920.79.
74.	157	(1) 1039.95, 874.25, 1107.88; (2) 5.302%, 3.522%, 3.066%; (3[a]) 14.540, .52749; (3[b]) 15.228, .58127; (3[c]) 15.775, .62527; (4[a]) 1072, 900, 1142.54; (4[b]) 1104.04, 925.74, 1177.19; (5[a]) 5.706%, 3.927%, 3.476%; (5[b]) 6.096%, 4.700%, 3.861%; (6) 1045.15, 880.81, 1112.43.
75.	159	(1) 457,310, 437,990, 515,519; (2) 541,537, 462,413, 428,253, 397,444; (3) 556,910, 450,870, 408,240, 371,100; (4) 907,572, 912,620, 1,107,287, 956,260; (5) 92,540, 108,407, 100,484.
76.	161	(1) 2.156%, 2.954%; (2) 2.67%, 3.12%, 3.5%; (3) 2.290%, 2.866%, 3.363%.
77.	163	(1) 10,484, 10,576; (2) 1.165%, 2.306%; (3) 365.47; (4) 10,933.30, 10,998.80; (5) 1.234%, 2.964%; (6) 27, 288.07, 9.35; (7) 11,226.50, 11,274.70; (8) 1.32%, 3.76%; (9) 13,860; (10) 16,174.20, 17,509.80; (11) 11,202.10, 10,479.30; (12) 1.313%, 3.685%; (13) 10,589; (14) 2.475%; (15) 4387.16.
78.	166	(1) 16,19; (2) $\frac{1}{4}$, $\frac{1}{8}$; (3) 25,36; (4) $\frac{1}{11}$, $\frac{1}{13}$; (5) -7, -10; (6) .016, .0032; (7) $\frac{1}{720}$, $\frac{1}{5040}$; (8) 4, 7; (9) $40\frac{1}{2}$, $60\frac{3}{4}$; (10) $\frac{2}{7}$, $\frac{2}{9}$.
79.	167	(1) 34,205; 29,216; 2, 27; $11\frac{2}{3}$, 280; 6, -7; 2 or 6, -5 or $+3$; (2) 13.75; (3) 3075; (4) 9500; (5) 2080, 1024, 1056, 260.
80.	169	(1) 1536, 3069; .0024576; .9983616; $\frac{1}{512}$, $\frac{1023}{512}$; 3, 728; 3, 6; $\sqrt{2}$, 5; 48, $93 + 45\sqrt{2}$.

EXERCISE	PAGE	
81.	170	(1) 5, 4, $\frac{16}{5}$; (3) $3\frac{1}{2}$, 5, $6\frac{1}{2}$; $2\sqrt{2}$, 4, $4\sqrt{2}$; $\frac{32}{13}$, $\frac{32}{10}$, $\frac{32}{7}$; (4) 5, 10, 15.
82.	170	(1) 4; (2) $\frac{4}{3}$; (3) 16; (4) $\frac{16}{7}$; (5) $2+\sqrt{2}$; (6) $2\sqrt{3}+\sqrt{6}$.
84.	174	(1) 739.70; (2) 1303.31; (3) 1926.68; (4) 741.92; (5) 519.03; (6) 1970; (7) 1347.85; (8) 507.61; (9) 1351.90; (10) 764.56.
86.	182	(1) 61,070; (2) 2.63%; (3) 449.30; (4) 17.33%; (5) 2.5%; (6) 844.70; (7) 45.41; (8) 1491.80, 548.80, 1632, 4069.
87.	188	(2) 154.17, 23.72, 19.4%; (4) 205.60, 15.3, 1216, 10.18%, 205.89, 1213.98; (4[g]) 100, 90, 81, 27.43, 24.69, 22.22; (6) 400.70.
88.	191	(1) 196.80; (3) 256.80.
89.	193	(1[a]) By 1870.55 − 1766.67; (2[b]) by 1766.67 − 1442.95; (3) yrs.; (4) 7.84; (5) 55,007; (6) 40,456; (7) 10,412; (8) 3.23%; (9) $1.47; (10) 3.9%; (11) 16.46; (12) 3358.20; (13) 14.58.
91.	202	(3) 215.83, 381.27, 3166.90, 6161.70; (5) 1514.70; (6) 351.61.
92.	205	(1) 166.53, 3233, 9336, 8976; (2[a]) 1, 0; (2[b]) 213; (2[c]) 184, 456, 566; (2[d]) 765, 1000; (3) 1064, 1019, 1000.
93.	213	(1) $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{2}{5}$; (2) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{13}$, $\frac{1}{52}$, $\frac{12}{13}$, $\frac{3}{4}$, $\frac{10}{13}$; (3) $\frac{1}{6}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{3}$; (4) $\frac{1}{7}$, $\frac{6}{7}$, $\frac{2}{7}$; (5) $\frac{1}{9}$, $\frac{8}{9}$.
94.	215	(1) $\frac{3}{5}$, $\frac{2}{5}$, 1; (2) $\frac{1}{13}$(6, 4, 3, 7, 9, 10, 10, 3); (3) $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{4}$, $\frac{5}{8}$, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{5}{8}$, 1; (4) $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{5}$, 0.14, 0.1, 0.2; (5) 0.3, 0.2, $\frac{9}{65}$, $\frac{8}{65}$; (6) 0.936, 0.064; (7) 0.936, 0.648, 0.216; (8) $\frac{5}{16}$, $\frac{11}{16}$, $\frac{15}{16}$; (9) $\frac{16}{27}$, $\frac{8}{9}$, $\frac{80}{81}$; (11) $\frac{1}{16}$, $\frac{15}{16}$; (13) 2 to 1; (14) 6 to 5, 3 to 5, 63 to 25, 96 to 25; (15) 2 to 1, 1 to 1, 5 to 1, 8 to 1; (16) 3 to 2, 3 to 4, 27 to 8, 21 to 4; (17) 111 to 10; (19) 216, 18 and 3; (20) 28, 55; (21) $\frac{1}{7}$, $\frac{4}{55}$; (22) $\frac{1}{77}$.

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97.	221	(1) 20, 120, 120, 720, 504, 336; (2) 20, 120, 90, 504, 8, 210; (4) $52! \div 44!$, $20! \div 10!$; (5) 1.29×10^{58}.
99.	225	(1) $\frac{1}{222}$, $\frac{1}{66}$, $\frac{10}{38}$, $\frac{2}{11}$, $\frac{5}{11}$; (2) 3360, 1920; (3) 60, 60, 35; (4) 21, 35, 2002.
100.	229	(1) $\frac{1}{1024}$ (1, 210, 252, 45, 848, 386); (2) $\frac{1}{7776}$ (1250, 250, 1, 1526, 7750); (3) $\frac{4}{7}\left(\frac{6}{7}\right)^6$, $1 - \frac{15}{7}\left(\frac{6}{7}\right)^7$, $\left(\frac{6}{7}\right)^8$; (4) approx. 0.20, 0.12, 0.25; (5) $\frac{2}{5}$ app., $\frac{5}{32}$, $\frac{1}{1024}$, $\frac{15}{64}$, $\frac{1}{32}$, $\frac{12}{48}$ app., $\frac{2}{48}$ app.; (6) approx. 0.235, 0.872, 0.042.
101.	235	(1) $3\Sigma x - 2n$; (2) $9\Sigma x^2 - 12\Sigma x + 4n$; (3) $\Sigma 4xy$; (4) $\Sigma x^2 + \Sigma y^2$; (5) $2\Sigma xy$; (6) $2\Sigma x + 3\Sigma y - 4n$; (7) $\Sigma x^2 - \Sigma x$; (8) $\Sigma x^2 - \Sigma x - 6n$; (9) $\Sigma x^2 + 2\Sigma xy + \Sigma y^2 + 2\Sigma x + 2\Sigma y + n$; (10) $4\Sigma x^2 + 12\Sigma xy + 9\Sigma y^2 - 4\Sigma x - 6\Sigma y + n$.
102.	237	(3) 80; (4) 72.7; (5) 7.
103.	238	(1) 50; (2) 2; (3) 40, 80, 100; (4) 360, 320, 300; (5) $\frac{2}{5}$, $\frac{3}{5}$, 40; (6) 50, $\frac{2}{5}$, $\frac{3}{5}$.
104.	241	(1) 7.46, 1.26, 2.42; (2) $\bar{x} = 5\frac{14}{15}$, $\sigma = .93$.
105.	243	(1) 50, 5; (2) 60, $\sqrt{50}$; (3) 240, $\sqrt{80}$; (4) $\sigma = \frac{5}{3}\sqrt{5}$; (5) $\sigma = \sqrt{240}$.
106.	249	(1) 17.326; (2) 11.2296; (3) 0.54256; (4) 6.3944; (5) 0.8654; (6) 2.4995.
107.	252	(1) 0.3413, 0.1360, 0.0214, 0.0256; (2) ± 0.97, -0.36, 0.25; (3) 1.08%, 2.87%, 98.21%; (4) 4.84%, 81.59%; (5) 3.76%, 95.51%, 8.25%; (6) 2.4%, 17.0%, 85.4%.
108.	256	(1) O, 45°; (2) Y; (3) X; (4) O; (5) X; (6) Y; (7) O, 45°; (8) 45°; (9) X; (10) X.
109.	257	(1) ± 3; (2) $x = 5$, no y intercept, x may not be less than 5; (3) y may not be less than 5; (4) intercepts are $x = 3, -1$ and $y = 3$, y may not be greater than 4; (5) intercepts are $y = -\frac{5}{3}$,

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109.	257	$x = -\frac{5}{2}$, asymptotes are $x = 3$, $y = 2$; (6) intercepts are $y = \pm\sqrt{2}$, y may not be between $-\frac{4}{3}$ and $+\frac{4}{3}$.
110.	259	(1) $x^2 + y^2 = 25$, $y = 6x$, $xy = 18$, $y^2 + 3x = 9$, $x^3 - x = y$; (3) new origin at: $(-3, 0)$, $(0, 2)$, $(3, -2)$, $(0, 2)$, $(-\frac{1}{2}, k)$, $(\frac{5}{4}, k)$, $(-2, 0)$, $(-h, \frac{3}{2})$, $(h, \frac{3}{2})$, $(0, 2)$, $(1, k)$, $(0, 0)$.
111.	265	(1) $2y - 3x = -5$; (2) $y - x = 2$; (3) $3y + 2x = 38$; (4) $5y - 4x = -5$; (5) $cy - dx = bc - ad$; (6) $y \log 2 - 4x = 3 \log 2 - 4 \log 5$; (8) $3^y = 2^x\ (0.6)^{\log 2}$; (9) $-\frac{7}{2}$, $-\frac{2}{3}$; (10) $x = \frac{1}{4} \log 1250$, $y = 14.23$.
112.	268	(1) $2y - 3x = 0$, $y = 3 \cdot 2^{\frac{1}{2}(x-2)}$, $x^3 = 2^y$, $2y - 3x = 0$; (2) $4y - x = 10$, $\left(\frac{y}{3}\right)^4 = \left(\frac{4}{3}\right)^{x-2}$, $2 \cdot 3^y = 27x$, $\left(\frac{y}{3}\right)^{\log 3} = \left(\frac{x}{2}\right)^{\log \frac{4}{3}}$; (3) $y - 4x = -10$, $y = 2 \cdot 3^{x-3}$, $\left(\frac{x}{3}\right)^4 = \left(\frac{4}{3}\right)^{y-2}$, $\left(\frac{y}{2}\right)^{\log \frac{4}{3}} = \left(\frac{x}{3}\right)^{\log 3}$; (4) $3y - 2x = 0$, $y^3 = 2^x$, $4x^2 = 9 \cdot 2^y$, $3y - 2x = 0$; (5) $y + 3x = 18$, $y = 3 \cdot 2^{5-x}$, $\left(\frac{x}{5}\right)^3 = \left(\frac{4}{5}\right)^{y-3}$, $\left(\frac{y}{3}\right)^{\log \frac{4}{5}} = \left(\frac{x}{5}\right)^{\log 2}$.
113.	270	(1) $3^{y+2} = x^4$, $p = 46.72$, $q = 8.09$; (2) $3^{x+2} = y^4$, $p = 12.24$, $q = 11.84$; (3) $2y = 3x$, $p = \frac{22}{3}$, $q = 18$; (4) $x = 3y$, $p = 24$, $q = 16\frac{2}{3}$; (5) $\left(\frac{y}{b}\right)^d = r^{x-a}$; (6) $\left(\frac{y}{3}\right)^6 = 2^{x-2}$, $p = 18.42$, $q = 13.47$; (7) $\left(\frac{3x}{2}\right)^7 = 3^{5y}$, $p = 7.02$, $q = 4.33$; (8) $y =$

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113.	270	$\log x$; (9) $\left(\frac{y}{5}\right)^{\log 2} = \left(\frac{x}{2}\right)^{\log 3}$, $p = 8.15$, $q = 177.65$; (10) $2y = 5x$, $p = 20$, $q = 50$; (11) $\left(\frac{y}{405}\right)^{\log 2} = \left(\frac{2}{x}\right)^{\log 3}$, $p = 7.165$, $q = 11.37$; (12) $xy = 360$, $p = 36$, $q = 6$; (13) $\left(\frac{y}{2}\right)^{\log 3} = \left(\frac{x}{3}\right)^{\log 5}$; (14) $\left(\frac{y}{100}\right)^{\log 3} = \left(\frac{2}{x}\right)^{\log 5}$, $y = \sqrt{80}$; (15) $2y - 5x = 5$, $2y^2 = 25\ (2^x)$, $3^y = (3x)^5$, $(.2y)^{\log 3} = x^{\log 2}$.
115.	277	(1) $xy = y - x + 19$; (2) $2x^2 = 3x + y - 2$; (3) $2y^2 = 3y - x + 5$; (4) $xy = 2y + 3x + 12$; (5) $xy = x - y - 19$; (6) $2y^2 = x + 3y - 2$; (7) $3x^2 = 3x + y - 2$; (8) $xy = 3x - 2y - 24$; (9) $xy = 3x + 2y + 12$.
116.	278	(1) $2x = 3(2^t)$, $y = t^3$; (2) $2x = 3(2^t)$, $y = t^2 + t + 5$; (3) $2y = 3t^2 - 3t + 10$, $x = t^2 + t + 5$; (4) $2x = 3(2^t)$, $3y = 5(3^t)$; (5) $x = (t+2)^2$, $3y = 2(3^t)$.
117.	279	(3) $y = 5x^{\frac{1}{2}} - 2$; $\left(\frac{y}{2}\right)^2 = 5^{x-3}$; $\left(\frac{x}{2}\right)^3 = 3^{y-3}$; $\left(\frac{y}{4}\right)^{\log 2} = \left(\frac{x}{5}\right)^{\log 3}$; $y = 2x^2 + 8x + 11$; $x = 2 \cdot 3^{t-1}$, $y = t^3$.
118.	285	(1) $\frac{5}{4}$, $-\frac{3}{2}$; (3) $\frac{21}{4}$, $\frac{17}{4}$, $\frac{21}{4}$, 6; (5[a]) $\frac{215}{164}$, $\frac{86}{115}$; (5[b]) $\frac{47}{185}$, $-\frac{261}{925}$.
121.	291	(2) $y = \frac{5}{7}x + 5$, $y = -\frac{19}{42}x^2 + \frac{110}{21}x - \frac{9}{2}$; $y = -\frac{1}{9}x^3 + \frac{17}{14}x^2 - \frac{146}{63}x + \frac{11}{2}$; (6) $y' = \frac{16}{15}x' - \frac{10}{9}$, $44\frac{28}{45}$; $y' = -\frac{10}{11} + \frac{16}{15}x' - \frac{1}{33}x'^2$, 44.3; $y' = -.91 - .56x' - .03x'^2 + .14x'^3$, 15.18.
122.	293	(1) $y' = 1.4x' - 1$, $x' = .7y' + .7$, 1.6, 0.8, (5, 8); (2) both lines are $2y - 3x = 1$, $\Sigma P^2 = 0$; (3) $y = 1.45x - .18$, $x = 66y + .31$, 2.08, .98,

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122.	293	(4.4, 6.2); (4) $y = 12.5 - 2.3x$, $x = 5.13 - .38y$, 2.3, 1.23, (3, 5.6); (5) $y = 29 - 2x$, $x = 14.37 - .49x$, 4, 1.7, (9, 11).
123.	297	(1) 84.33, 83.13, 85, 83.6; (2) 78.76, 79.43, 68.13, 70.64; (3) 3.75, 40.6, 35.54, .33, −40.49, 29.87; (4) 0.05, 0.56, 0.48, 0.004, −0.54, .43; (5) regression equations for yw are $y - 83.13 = -\frac{0.54}{70.64}(w - 83.6)$ and $w - 83.6 = -\frac{0.54}{79.43}(y - 83.13)$, and similarly for the remaining pairs.
124.	300	(1) (3.01, −0.21), 1.05; (2) (1.60, 0.49), 31.24; (3) (2.55, 1.29), 21.74; (4) (1.58, 2.60), 9.56; (5) (1.18, 3.02), 2.36; (6) (2.78, 0.50), 59.4; (7) (1.23, 0.54, 0.82), 26.43.

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